Investigating Traces of Matrix Products

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by

Andrew Schneider

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Advisor: John Greene

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Dedication

To my Mother and Father, for providing nothing but support, praise and love throughout my entire life.
Abstract

The trace of a square matrix $A$ is the sum of the diagonal entries in $A$, and is denoted $Tr(A)$. In this paper we investigate the relative size of the trace of a product of matrices. We consider how both the ordering of the product and the number of matrices in the product influences the size of the trace. Data was collected for products of real-valued matrices with independent random variable entries from a standard normal distribution. We first considered two $n \times n$ matrices $A$ and $B$ and compared $Tr(ABAB)$ vs $Tr(AABB)$ as $n$ increased. We then considered products of $2A$’s and $mB$’s for $2 \times 2$ matrices $A$ and $B$. When $m = 4$, there are three possible traces to consider, $1 = Tr(AB^2AB^2)$, $2 = Tr(ABAB^3)$, and $3 = Tr(A^2B^4)$. Here, all possible orderings of the traces were investigated and it was found that the permutation 231 did not occur. This investigation was extended to larger numbers of $B$’s, asking which permutations of the orders are possible and which are not.
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Chapter 1

Introduction

The trace of a square matrix $A$ is the sum of the diagonal entries in $A$, and is denoted $Tr(A)$ [7, p. 90]. As an example, if $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 5 & 4 & 2 & 5 \\ 8 & 6 & 6 & 2 \\ 3 & 6 & 9 & 0 \end{pmatrix}$, we find that $Tr(A) = 1 + 4 + 6 + 0 = 11$. When considering a product of matrices $A$ and $B$, we have two possible products to examine, $AB$ and $BA$. It turns out that if $A$ and $B$ are square matrices of the same size then $Tr(AB) = Tr(BA)$. For example, if $A = \begin{pmatrix} 2 & 1 & 8 \\ 3 & 4 & 3 \\ 1 & 6 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 5 \\ 8 & 1 & 2 \end{pmatrix}$, we have $AB = \begin{pmatrix} 69 & 13 & 21 \\ 34 & 23 & 26 \\ 64 & 37 & 44 \end{pmatrix}$, $BA = \begin{pmatrix} 4 & 2 & 16 \\ 22 & 51 & 58 \\ 21 & 24 & 81 \end{pmatrix}$ and $Tr(AB) = Tr(BA) = 136$.

Lemma 1.1. If $A$ and $B$ are square matrices of the same size then

$$Tr(AB) = Tr(BA)$$

Proof. Let $A = \begin{pmatrix} a_{ij} \end{pmatrix}$ and $B = \begin{pmatrix} b_{ij} \end{pmatrix}$ be $n \times n$ matrices. Then

$$Tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}a_{ij} = Tr(BA)$$
When considering the trace of a product of matrices, it is well known that the product of matrices is invariant under cyclic permutations\[7, p. 110\]. That is, for matrices $A, B$ and $C$, $Tr(ABC) = Tr(CAB) = Tr(BCA)$, where the products $CAB$ and $BCA$ are cyclic permutations of $ABC$. Thus, these three permutations are equivalent when considering their traces.

**Theorem 1.2.** Let $A_1, A_2, ..., A_n$ be $m \times m$ matrices. Then

$$Tr(A_1...A_{n-1}A_n) = Tr(A_nA_1...A_{n-1})$$

**Proof.** By Lemma 1.1 for $m \times m$ matrices $A_1$ and $A_2$, $Tr(A_1A_2) = Tr(A_2A_1)$.

In general, for matrices

$$A_1, A_2, ..., A_n$$

Let

$$B = A_1A_2...A_{n-1}$$

Then we find that

$$Tr(BA_n) = Tr(A_nB)$$

or

$$Tr(A_1A_2...A_n) = Tr(A_nA_1...A_{n-1})$$

This does not hold for more general permutations however. In general $Tr(ABC) \neq Tr(CBA)$. For example, consider $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. Then

$$ABC = \begin{pmatrix} 60 & 50 \\ 50 & 40 \end{pmatrix}, \quad CBA = \begin{pmatrix} 47 & 58 \\ 39 & 46 \end{pmatrix}$$

and so $Tr(ABC) = 100 \neq 93 = Tr(CBA)$.

Given a collection of matrices, we define a necklace as the set of all cyclically permuted products of the collection. For example, when considering a product of matrices containing two $A$’s and two $B$’s there are two necklaces to consider, $\{ABAB, BABA\}$ and
\{AABB, BAAB, BBAA, ABBA\}. For convenience we write the matrices of a necklace in lexicographic order and we let the first permutation of a necklace represent the entire set. Thus for two A’s and two B’s, we would say there are two necklaces represented by AABB and ABAB. For three A’s and three B’s there are four necklaces represented by AAABBB, AABABB, AABBAB and ABABAB. By Theorem 1.2 we can think of the traces as acting on necklaces of products, rather than on products.

In this paper we investigate the value of the trace of a product of matrices under certain conditions. We build on Huang’s work which served as the motivation behind this research. For example, for a product of mA’s and nB’s, as m and n increase, what happens to the trace of their product? For two n × n matrices A and B, how does increasing n affect the trace of their product? We considered which necklaces have the larger trace for different m and n values, however we almost exclusively restricted our simulations to the case of m = 2. We also investigated which traces can never be largest.

In Chapter 2, data for our initial simulations is presented. All of our simulations in this paper use real valued matrices with independent random variable entries from a standard normal distribution. We give the results for the products of two A’s and two B’s, where matrices A and B are n × n, for n ranging from 2 to 1000. We also present simulation results for a product of two A’s with two and three B’s, where A and B are 2 × 2, 3 × 3 and 4 × 4 matrices. In these simulations we considered the effects that complex eigenvalues have on the relative size of the trace. In the 2 × 2 case we explain most of the probabilities in our simulations for a product of two A’s with three B’s.

We then present more trace data in Chapter 3 and introduce the idea of a forbidden ordering. Chapter 4 is more theoretical. We introduce two variable Lucas Polynomials in order to derive a formula that relates a polynomial to the difference for the traces of a product of 2 × 2 matrices A and B. Using this formula we were able to verify that orders in our simulations are indeed forbidden. Suggestions for future work are contained within Chapter 5 and the appendix includes several samples of code used for collecting data in our simulations.
Chapter 2

Some Initial Data

We first considered a product of two $A$’s and two $B$’s, with independent random variable entries from a standard normal distribution. In general, $\text{Tr}(AABB) \neq \text{Tr}(ABAB)$ so we investigated how these traces compare for matrices of various sizes. Table 2.1 shows our results after 1,000,000 trials.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{Tr}(ABAB) &gt; \text{Tr}(AABB)$</th>
<th>$\text{Tr}(AABB) &gt; \text{Tr}(ABAB)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>707359</td>
<td>292641</td>
</tr>
<tr>
<td>3</td>
<td>703919</td>
<td>296081</td>
</tr>
<tr>
<td>4</td>
<td>701421</td>
<td>298579</td>
</tr>
<tr>
<td>5</td>
<td>701513</td>
<td>298487</td>
</tr>
<tr>
<td>10</td>
<td>705472</td>
<td>294528</td>
</tr>
<tr>
<td>20</td>
<td>710109</td>
<td>289891</td>
</tr>
<tr>
<td>30</td>
<td>713266</td>
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</tr>
<tr>
<td>40</td>
<td>714237</td>
<td>285763</td>
</tr>
<tr>
<td>50</td>
<td>714643</td>
<td>285357</td>
</tr>
<tr>
<td>100</td>
<td>716399</td>
<td>283601</td>
</tr>
<tr>
<td>1000</td>
<td>718481</td>
<td>281519</td>
</tr>
</tbody>
</table>

Table 2.1: Simulations on $\text{Tr}(ABAB)$ vs $\text{Tr}(AABB)$ for $n \times n$ matrices as $n$ varies.
Dr Greene [4] had done some research on products of matrices and found that for $2 \times 2$ matrices $A$ and $B$, $Tr(ABAB) > Tr(AABB)$ with probability $\frac{1}{\sqrt{2}}$. This explains our results for $n = 2$ in Table 2.1 as $\frac{1}{\sqrt{2}} \approx 0.707$, so one expects roughly 707,000 vs 293,000. As we increased the size of matrices $A$ and $B$, this probability decreased initially and then slowly began to increase. The probability that $Tr(ABAB) > Tr(AABB)$ appears to change slowly as the size of the matrices, $n$, increases. As Table 2.1 indicates, there are no sudden jumps from $n = 50$ to $n = 100$ or $n = 1000$, it appears to be fairly stable for larger $n$.

During our investigation we considered how complex eigenvalues might affect these traces. By Lemma 3.7 in [4], if matrix $A$ has independent normally distributed elements of mean 0 and variance 1, then the probability that $A$ has real eigenvalues is $\frac{1}{\sqrt{2}}$. Another result from Dr Greene [4] is that $Tr(ABAB) > Tr(A^2B^2)$ whenever either $A$ or $B$ has complex eigenvalues. We first considered when $A$ and $B$ are $2 \times 2$ matrices, as shown in Table 2.2 again with 1,000,000 trials. We use $r$ to indicate the given matrix having real eigenvalues and $c$ to indicate the given matrix having complex eigenvalues.

<table>
<thead>
<tr>
<th>A eigenvalues</th>
<th>r</th>
<th>r</th>
<th>r</th>
<th>r</th>
<th>c</th>
<th>c</th>
<th>c</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>B eigenvalues</td>
<td>r</td>
<td>r</td>
<td>c</td>
<td>c</td>
<td>r</td>
<td>r</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>$Tr(ABAB) &gt; Tr(A^2B^2)$</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Count</td>
<td>292544</td>
<td>207597</td>
<td>0</td>
<td>207510</td>
<td>0</td>
<td>206602</td>
<td>0</td>
<td>85747</td>
</tr>
</tbody>
</table>

Table 2.2: Eigenvalues for $2 \times 2$ products of $2A$’s with $2B$’s.

These results can all be explained. We know that $Tr(ABAB)$ has the larger trace when either $A$ or $B$ has complex eigenvalues, thus explaining why the counts for columns three, five and seven are 0. Column one counts all the cases where $Tr(ABAB) < Tr(AABB)$, which is $(1 - \frac{1}{\sqrt{2}}) \approx 0.292$. Column eight counts the cases when $A$ and $B$ have complex eigenvalues, which happens with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{3}{2} - \sqrt{2} \approx 0.085$. Now, columns four and six are the sum of total cases where one matrix has real eigenvalues and the other has complex. This happens with probability $\frac{1}{\sqrt{2}}(1 - \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2} \approx 0.207$. Finally, we know that $Tr(ABAB) > Tr(AABB)$ with probability $\frac{1}{\sqrt{2}}$, so column two
occurs with probability $\frac{1}{\sqrt{2}} - (2\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \left(\frac{3}{2} - \sqrt{2}\right)) \approx 0.207$.

We also considered when $A$ and $B$ are $n \times n$ matrices, for $n = 3$ and $n = 4$, as shown in Table 2.3 and Table 2.4 respectively. We again ran our simulations for 1,000,000 trials. A $3 \times 3$ matrix will always have at least one real eigenvalue because any complex eigenvalues occur in conjugate pairs. As such, the columns in Table 2.3 labeled $c$ correspond to that given matrix having one real and two complex eigenvalues.

<table>
<thead>
<tr>
<th>A eigenvalues</th>
<th>r</th>
<th>r</th>
<th>r</th>
<th>c</th>
<th>c</th>
<th>c</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>B eigenvalues</td>
<td>r</td>
<td>r</td>
<td>c</td>
<td>c</td>
<td>r</td>
<td>r</td>
<td>c</td>
</tr>
<tr>
<td>$Tr(ABAB) &gt; Tr(A^2B^2)$</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Count</td>
<td>77386</td>
<td>47657</td>
<td>64586</td>
<td>164094</td>
<td>64326</td>
<td>164732</td>
<td>90363</td>
</tr>
</tbody>
</table>

Table 2.3: Eigenvalues for $3 \times 3$ products of $2A$’s with $2B$’s.

We can explain some features of columns 3, 4, 5 and 6 using symmetry. Consider the entries in columns 3 and 5 for example, since we consider a product of two $A$’s and two $B$’s, the probability of $A$ real $B$ complex should equal $A$ complex $B$ real. Thus we can explain the cases when $Tr(A^2B^2) > Tr(ABAB)$. Likewise, the yes cases corresponding to columns 4 and 6 should have equal probabilities, which are observed in our simulations.

It was shown in [3] that if $A$ is a $3 \times 3$ matrix, then three real eigenvalues should occur with probability $\frac{\sqrt{2}}{4}$ and one real eigenvalue should occur with probability $1 - \frac{\sqrt{2}}{4}$. In Table 2.3, if we consider the cases when matrix $A$ has real eigenvalues, this occurred in $77386 + 47657 + 64586 + 164094 = 353723$ of 1,000,000 trials and fits nicely with $\frac{\sqrt{2}}{4} \approx 0.353553$. The remaining cases account for $A$ having two complex and one real eigenvalues occurred $64326 + 164732 + 90363 + 326856 = 646277$ and again fits nicely with $1 - \frac{\sqrt{2}}{4} \approx 0.646447$.

With $4 \times 4$ matrices, the cases to consider are no real eigenvalues, two real eigenvalues or four real eigenvalues. The columns in Table 2.4 labeled 2c correspond to that matrix having two real and two complex eigenvalues. Similarly, the columns labeled 4r and 4c
correspond to four real or four complex eigenvalues, respectively.

<table>
<thead>
<tr>
<th>A eigenvalues</th>
<th>B eigenvalues</th>
<th>$Tr(ABAB) &gt; Tr(A^2B^2)$</th>
<th>Count</th>
</tr>
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<tr>
<td>4r</td>
<td>4r</td>
<td>no</td>
<td>10393</td>
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<td>4r</td>
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<td>yes</td>
<td>5271</td>
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<td>55621</td>
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<td>2c</td>
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<td>2c</td>
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<td>2c</td>
<td>4c</td>
<td>no</td>
<td>26145</td>
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<td>4c</td>
<td>4c</td>
<td>no</td>
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</tr>
<tr>
<td>4c</td>
<td>4c</td>
<td>yes</td>
<td>17839</td>
</tr>
</tbody>
</table>

Table 2.4: Eigenvalues for $4 \times 4$ products of $2A$’s with $2B$’s.

It was also shown in [3] that a $4 \times 4$ matrix $A$ has four real eigenvalues with probability $\frac{1}{8}$, two real eigenvalues with probability $-\frac{1}{4} + 11\sqrt{2}/16$ and no real eigenvalues with probability $\frac{9}{8} - 11\sqrt{2}/16$. From Table 2.4, consider those cases when matrix $A$ has four real eigenvalues. These occurred $10393 + 5271 + 34885 + 55621 + 4845 + 14395 = 125410$ from 1,000,000 trials, in close agreement with $\frac{1}{8} = 0.125$. Next, the cases when $A$ has two real and two complex eigenvalues occurred $34591 + 55706 + 151412 + 369583 + 26145 + 84145 = 721582$, closely matching $-\frac{1}{4} + 11\sqrt{2}/16 \approx 0.722272$. The remaining cases account for $A$ having no
real eigenvalues and occurred $4650 + 14250 + 26065 + 84794 + 5410 + 17839 = 153008$ agreeing with $\frac{9}{8} - 11 \frac{17}{16} \approx 0.152728$.

We then considered the case of $2 \times 2$ matrices again but with the addition of another matrix $B$ and compared $Tr(AABBB)$ vs $Tr(ABABB)$. With two $A$’s and an odd number of $B$’s we immediately noticed apparent symmetry. There is obvious symmetry in Table 2.5 among the yes and no values. The reason is that any involution on $A$ and $B$ which leaves them with standard normal variables should keep probabilities the same. An involution $f$ is a map which, if done twice, gets you back where you started. As an example consider $f(x) = -x$, then $f(f(x)) = x$ as desired. The relevant involution for our work is $f(B) = -B$. This means that the probability that $Tr(ABAB^2) > Tr(A^2B^3)$ is the same as the probability that $Tr(A(-B)A(-B)^2) > Tr(A^2(-B)^3)$. But this simplifies to $-Tr(ABAB^2) > -Tr(A^2B^3)$, or $Tr(A^2B^3) > Tr(ABAB^2)$. Now, since $B$ and $-B$ either both have real eigenvalues or both have complex eigenvalues, this involutions proves that the yes-no combinations occur with the same probability. In Table 2.5 $Tr(ABAB^2) > Tr(A^2B^3)$ in 500,549 of the 1,000,000 cases.

<table>
<thead>
<tr>
<th>A eigenvalues</th>
<th>r</th>
<th>r</th>
<th>r</th>
<th>r</th>
<th>c</th>
<th>c</th>
<th>c</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>B eigenvalues</td>
<td>r</td>
<td>r</td>
<td>c</td>
<td>c</td>
<td>r</td>
<td>r</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>$Tr(ABAB^2) &gt; Tr(A^2B^3)$</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
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<td>103557</td>
<td>103199</td>
<td>104059</td>
<td>42677</td>
<td>42789</td>
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</tbody>
</table>

Table 2.5: Eigenvalues for $2 \times 2$ products of 2A’s with 3B’s.

To explain these results in more detail we need to use the following.

**Lemma 2.1.** For any $2 \times 2$ matrix $M$,

$$M^2 = Tr(M)M - Det(M)I$$

where $I$ is the $2 \times 2$ identity matrix.

**Proof.** Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $M^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$
Now, \( \text{Tr}(M)M = \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} \) and \( \text{Det}(M)I = \begin{pmatrix} -bc + ad & 0 \\ 0 & -bc + ad \end{pmatrix} \).

So we have \( \text{Tr}(M)M - \text{Det}(M)I = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} \) as desired.

When we consider \( \text{Tr}(ABAB^2) > \text{Tr}(A^2B^3) \), this implies \( \text{Tr}(ABAB^2) - \text{Tr}(A^2B^3) > 0 \).

By Lemma 2.1, \( ABAB^2 = ABA[\text{Tr}(B)B - \text{Det}(B)I] = \text{Tr}(B)ABAB - \text{Det}(B)ABA \), and \( A^2B^3 = AAB[\text{Tr}(B)B - \text{Det}(B)I] = \text{Tr}(B)A^2B^2 - \text{Det}(B)A^2B \).

Thus

\[
\text{Tr}(ABAB^2) - \text{Tr}(A^2B^3) = \text{Tr}(B)(\text{Tr}(ABAB) - \text{Tr}(AABB))
\]

as \( \text{Tr}(ABA) = \text{Tr}(A^2B) \). From our involution, \( \text{Tr}(B) > 0 \) has exactly the same probability that \( \text{Tr}(-B) > 0 \). As a result \( \text{Tr}(B) > 0 \) with probability exactly \( \frac{1}{2} \). So if we add the yes-no entries we should get the probability of the combination of the eigenvalues.

We can use this information to explain the results in Table 2.3. Column one and two count the cases when \( A \) and \( B \) both have real eigenvalues, which happens with probability \( \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \). Since the yes-no possibilities are evenly divided, the probability of each will be 0.25. Next, column three and four count the cases when \( A \) has real eigenvalues and \( B \) has complex eigenvalues, which happens with probability \( \frac{1}{\sqrt{2}}(1 - \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2} \). Similarly, column five and six count the cases when \( A \) has complex eigenvalues and \( B \) has real eigenvalues, which happens with probability \( (1 - \frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{2} \). As a result, each of these four probabilities should be \( \frac{1}{2}(\frac{1}{\sqrt{2}} - \frac{1}{2}) \approx 0.103 \). Finally, column seven and eight count the cases when \( A \) and \( B \) both have complex eigenvalues, which happens with probability \( (1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{3}{2} - \sqrt{2} \). Since the yes-no possibilities are evenly divided, the probability of each will be \( \frac{1}{2}(\frac{3}{2} - \sqrt{2}) \approx 0.042 \).

This analysis becomes more and more complicated as the number of \( A \)'s and \( B \)'s increases. For example, consider the case of two \( A \)'s and four \( B \)'s. Now there are three necklaces denoted by \( M_1 = AB^2AB^2, M_2 = ABAB^3 \) and \( M_3 = A^2B^4 \). If we ask when \( \text{Tr}(AB^2AB^2) > \text{Tr}(A^2B^4) \), we get Table 2.6.
which is very much like Table 2.2. Dr Greene [4] found that if \( M \) is a product of \( A \)'s and \( B \)'s, then \( Tr(M^2) > Tr(MM^R) \) if and only if \( Tr(ABAB) > Tr(A^2B^2) \), where \( M^R \) is the reversal of \( M \), the product in reverse order. For example, if \( M = M_1M_2...M_n \) is a product of matrices, then the reversal of this product is defined as \( M^R = M_nM_{n-1}...M_1 \). Using this result for \( M = AB^2 \), this means that \( Tr(AB^2AB^2) > Tr(ABAB) = Tr(A^2B^4) \) with probability \( \frac{1}{\sqrt{2}} \), explaining why Table 2.6 mimics Table 2.2.

However, if we ask when \( Tr(ABAB^3) > Tr(A^2B^4) \) or when \( Tr(AB^2AB^2) > Tr(ABAB^3) \) we obtain significantly different tables, Table 2.7 and Table 2.8.

We first focus on Table 2.7. To explain these results in more detail we need to use the following.
By Lemma 2.1, $ABAB^3 = Tr(B)ABAB^2 - Det(B)ABAB$, and $AB^2AB^2 = Tr(B)ABAB^2 - Det(B)A^2B^2$. Thus

$$Tr(ABAB^2) - Tr(A^2B^3) = -Det(B)(Tr(A^2B^2) - Tr(ABAB))$$

$$= Det(B)(Tr(ABAB) - Tr(AABB))$$

Therefore we need information about $Det(B)$. The involution $f(B) = -B$ does not affect the determinant, however, the involution where two rows of $B$ are interchanged will change the sign of the determinant. This means $Det(B) > 0$ with probability $\frac{1}{2}$.

**Lemma 2.2.** [6, p. 288] Let $\lambda_1, \lambda_2, ..., \lambda_n$ be all the eigenvalues of an $n \times n$ matrix $M$. Then

$$Det(M) = \lambda_1\lambda_2...\lambda_n$$

$$Tr(M) = \lambda_1 + \lambda_2 + ... + \lambda_n$$

As a consequence, if $B$ is a $2 \times 2$ matrix with complex eigenvalues $c + di$ and $c - di$ then by Lemma 2.2, $Det(B) = (c + di)(c - di) = c^2 + d^2 > 0$. So $B$ will always have a positive determinant when it has complex eigenvalues. Since $Det(B) > 0$ half of the time, and all of the time when $B$ has complex eigenvalues, and $B$ has complex eigenvalues with probability $1 - \frac{1}{\sqrt{2}}$, it follows that the probability that $B$ has real eigenvalues and $Det(B) > 0$ is $\frac{1}{2} - (1 - \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2}$. Finally, since $B$ has real eigenvalues with probability $\frac{1}{\sqrt{2}}$, we need $p(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2}$, or $p = 1 - \frac{1}{\sqrt{2}}$.

What we have determined then is

$$Det(B) > 0 \quad \text{with probability} \quad \begin{cases} 1, & \text{if } B \text{ has complex eigenvalues}, \\ 1 - \frac{1}{\sqrt{2}}, & \text{if } B \text{ has real eigenvalues}. \end{cases}$$

and

$$Det(B) < 0 \quad \text{with probability} \quad \begin{cases} 0, & \text{if } B \text{ has complex eigenvalues}, \\ \frac{1}{\sqrt{2}}, & \text{if } B \text{ has real eigenvalues}. \end{cases}$$

It is useful to have a similar calculation for when $Tr(ABAB) > Tr(A^2B^2)$. We have to consider both $A$ and $B$ so there are four cases, depending on whether $A$ and $B$ have
real or complex eigenvalues. We know that $\text{Tr}(ABAB)$ has the larger trace when either $A$ or $B$ has complex eigenvalues. For real eigenvalues, the probability that $A$ has real eigenvalues and $B$ has real eigenvalues and $\text{Tr}(ABAB) - \text{Tr}(A^2B^2) > 0$ is $\frac{1}{\sqrt{2}} - \frac{1}{2}$. So we need $p\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \frac{1}{2}$, or $p = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = \sqrt{2} - 1$.

Therefore we have determined that

$$\text{Tr}(ABAB) > \text{Tr}(A^2B^2) \quad \text{with probability } \begin{cases} 1, & \text{if complex, complex,} \\ 1, & \text{if complex, real,} \\ 1, & \text{if real, complex,} \\ \sqrt{2} - 1, & \text{if real, real eigenvalues.} \end{cases}$$

With this information we can explain most of Table 2.8. The columns are linked in pairs, in other words, the sum of columns 1 and 2, 3 and 4, 5 and 6, 7 and 8 is the probability of a particular eigenvalue combination.

First, columns 1 and 2 count the cases when $A$ and $B$ both have real eigenvalues, this occurs with probability $\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$, which fits nicely with our data, $211671 + 287999 = 499670 \approx 500000$. Next, Columns 3 and 4 count the cases when $A$ has real and $B$ has complex eigenvalues, this occurs with probability $\left(\frac{1}{\sqrt{2}}\right)(1 - \frac{1}{\sqrt{2}}) = \frac{1}{2}(\sqrt{2} - 1) \approx 0.207$, which matches our data, $0 + 207536 \approx 207000$. Columns 5 and 6 count the cases when $A$ has complex and $B$ has real eigenvalues, this occurs with probability $(1 - \frac{1}{\sqrt{2}})\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}(\sqrt{2} - 1) \approx 0.207$, matching $146513 + 60857 = 207370 \approx 207000$. Finally, Columns 7 and 8 count the cases when $A$ and $B$ both have complex eigenvalues, this occurs with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{1}{2}(3 - 2\sqrt{2}) \approx 0.085$, again matching our data $0 + 85424 = 85424 \approx 85000$.

Since the columns are linked if we explain one linked column we will explain the other by default. For example, column 8 counts the case when $A$ has complex eigenvalues and $B$ has complex eigenvalues and $\text{Tr}(ABAB) - \text{Tr}(A^2B^2) > 0$, this happens with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}})(1) = \frac{1}{2}(3 - 2\sqrt{2}) \approx 0.085$. Now, since we know columns 7 and 8 add to 85,000, column 7 must be 0. In fact, we already knew this because $\text{Tr}(ABAB) - \text{Tr}(A^2B^2) < 0$ with probability 0 when $B$ has complex eigenvalues, and
$Det(B) > 0$ with probability 1 when $B$ has complex eigenvalues.

Similarly column 3 must be 0 since $B$ has complex eigenvalues and $Tr(ABAB) - Tr(A^2B^2) < 0$ with probability 0 when in this case. Thus, column 4 must be $\approx 0.207$, matching our data nicely, 207536. Finally, column 6 counts the case when $A$ has complex eigenvalues and $B$ has real eigenvalues and $Tr(ABAB) - Tr(A^2B^2) > 0$, which means we need $Det(B) > 0$ as well. This occurs with probability $(1 - \frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{1}{4}(3\sqrt{2} - 4) \approx 0.0607$, which is close to 60587. Now that we know column 6, column 5 must be $\frac{1}{2}(\sqrt{2} - 1) - \frac{1}{4}(3\sqrt{2} - 4) = \frac{1}{4}(2 - \sqrt{2}) \approx 0.146$, matching 146536 nicely. We have explained columns 3, 4, 5, 6, 7 and 8. However, we were unable to explain columns 1 and 2.

The same analysis done in Table 2.8 will work in Table 2.7, except it is a bit more complicated. Since $Tr(ABAB^3) - Tr(A^2B^4) = (Tr(B)^2 - Det(B))(Tr(ABAB) - Tr(A^2B^2))$, things depend also on the sign of $Tr(B)^2 - Det(B)$. However, some features of Table 2.7 can be explained. If $B$ is a $2 \times 2$ matrix with real eigenvalues $\lambda_1$ and $\lambda_2$ then by Lemma 2.2, $Tr(B)^2 - Det(B) = (\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2 = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2$. Since this is always nonnegative, when $B$ has real eigenvalues, the sign of $Tr(ABAB^3) - Tr(A^2B^4)$ matches the sign of $Tr(ABAB) - Tr(A^2B^2)$. This means that columns where $B$ has real eigenvalues should match the corresponding columns in Tables 2.6 and 2.7. This explains why columns 1, 2, 5 and 6 are equal in both Tables.

In general, if $M_1, M_2$ are each products of $mA$’s and $nB$’s, based on the involution $Tr(M_1) > Tr(M_2)$ with probability $\frac{1}{2}$ if either $m$ or $n$ is odd. Thus, the most interesting case is when $m$ and $n$ are both even.
Chapter 3

More Trace Data

In this chapter we investigate the frequencies of the possible orderings for products of matrices with $mA$’s and $nB$’s. For $2A$’s and $4B$’s there are three necklaces and we first investigated the frequencies of the 6 possible orderings of these necklaces.

Table 3.1 shows the count of each necklace when $m = 2$ and $n = 4$ and 5. We use numbers to represent the possible orderings. In general, with $2A$’s and $nB$’s, necklace 1 represents the necklace with the $A$’s separated (cyclically) as far as possible. If $n = 2m$, necklace 1 would be $AB^mAB^m$ and the $k^{th}$ necklace would be represented by $AB^{m-k+1}AB^{m+k-1}$. If $n = 2m + 1$, necklace 1 would be $AB^mAB^{m+1}$ and the $k^{th}$ necklace would be represented by $AB^{m-k+1}AB^{m+k}$.

As an example, with two $A$’s and four $B$’s let the ordering 123 correspond to $Tr(AB^2AB^2) > Tr(ABAB^3) > Tr(A^2B^4)$, and a count of 300975 means the the ordering $Tr(AB^2AB^2) > Tr(ABAB^3) > Tr(A^2B^4)$ occurs 300975 out of 1,000,000 trials.
Table 3.1: Initial Data for $2 \times 2$ products containing 2A’s and 4−5B’s.

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<th>2A4B</th>
<th>2A5B</th>
</tr>
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<td>Count</td>
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</tr>
<tr>
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<td>281835</td>
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<tr>
<td>231</td>
<td>0</td>
</tr>
<tr>
<td>312</td>
<td>217850</td>
</tr>
<tr>
<td>321</td>
<td>75780</td>
</tr>
<tr>
<td>5 permutations</td>
<td>6 permutations</td>
</tr>
</tbody>
</table>

1 = ABBABB 1 = ABBABB |
2 = ABABBB 2 = ABABBB |
3 = AABBBB 3 = AABBBB |

Notice that for 2A’s and four B’s the ordering 231 is, at best, very unlikely to occur, yet with 2A’s and five B’s the order 231 occurs with probability $\approx 0.029$. We call an order that is not possible a forbidden order.

Table 3.2 lists our simulation results for $m = 2$, $n = 6, 7, 8$ and 9. We only list those orderings that occurred a positive number of times in a sample size of 1,000,000 trials. Notice that of the 4! possible orderings that could occur for two A’s and six B’s, only eight appear to be occurring. Similarly, only twelve of the twenty four possible orderings are occurring for two A’s and seven B’s. As we increase the number of B’s, the proportion of the number of possible orderings appears to significantly decrease.
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Table 3.2: 2 × 2 products products containing 2A’s and 6 – 9B’s.
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<th>Ordering</th>
<th>Count</th>
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1 = AB^5AB^5  
2 = AB^4AB^6  
3 = AB^3AB^7  
4 = AB^2AB^8  
5 = ABAB^9    
6 = A^2B^{10}  

Table 3.3: 2 × 2 products containing 2A’s and 10 − 11B’s.
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<td>8964</td>
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<td>1562473</td>
<td>18048</td>
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<td>1634527</td>
<td>21247</td>
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<td>1643725</td>
<td>18735</td>
</tr>
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<td>1652743</td>
<td>14186</td>
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<td>1672534</td>
<td>10979</td>
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<td>1726354</td>
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<td>1762534</td>
<td>8727</td>
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<td>6421357</td>
<td>282070</td>
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<tr>
<td>7531246</td>
<td>217631</td>
</tr>
</tbody>
</table>

Table 3.4: $2 \times 2$ products containing $2A$'s and $12-13B$'s.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
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<tbody>
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<td>Count</td>
<td>Ordering</td>
<td>Count</td>
<td>Ordering</td>
<td>Count</td>
<td>Ordering</td>
<td>Count</td>
<td></td>
</tr>
<tr>
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<td>500326</td>
<td>123</td>
<td>74577</td>
<td>123</td>
<td>177022</td>
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<td>703339</td>
<td>21</td>
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<td>132</td>
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<td>132</td>
<td>230613</td>
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<tr>
<td></td>
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<td>57709</td>
<td>213</td>
<td>92866</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>231</td>
<td>231017</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
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<td>312</td>
<td>91911</td>
<td></td>
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<td></td>
<td></td>
</tr>
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<td>176571</td>
<td></td>
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<td></td>
<td></td>
</tr>
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<td>2 permutations</td>
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<td>6 permutations</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>1 = AABB</td>
<td>1 = AABB</td>
<td>1 = AABB</td>
<td>1 = AABB</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 = ABAB</td>
<td>2 = ABAB</td>
<td>2 = ABAB</td>
<td>2 = ABAB</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 = ABBABB</td>
<td>3 = ABBABB</td>
<td>3 = ABBABB</td>
<td>3 = ABBABB</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.5: $3 \times 3$ products products containing 2A’s and 2 – 5B’s.
<table>
<thead>
<tr>
<th>2A6B</th>
<th>2A7B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordering</td>
<td>Count</td>
</tr>
<tr>
<td>1234</td>
<td>26553</td>
</tr>
<tr>
<td>1243</td>
<td>28380</td>
</tr>
<tr>
<td>1324</td>
<td>10030</td>
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<tr>
<td>1342</td>
<td>79288</td>
</tr>
<tr>
<td>1423</td>
<td>35135</td>
</tr>
<tr>
<td>1432</td>
<td>52934</td>
</tr>
<tr>
<td>2134</td>
<td>8269</td>
</tr>
<tr>
<td>2143</td>
<td>10482</td>
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<tr>
<td>2314</td>
<td>5343</td>
</tr>
<tr>
<td>2341</td>
<td>43015</td>
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<td>2413</td>
<td>35209</td>
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<td>2431</td>
<td>117052</td>
</tr>
<tr>
<td>3124</td>
<td>11664</td>
</tr>
<tr>
<td>3142</td>
<td>34169</td>
</tr>
<tr>
<td>3214</td>
<td>3461</td>
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<tr>
<td>3241</td>
<td>25662</td>
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<td>3412</td>
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<td>3421</td>
<td>49502</td>
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<td>4123</td>
<td>42170</td>
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<td>4132</td>
<td>56190</td>
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<td>4213</td>
<td>56418</td>
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<td>52518</td>
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<td>4321</td>
<td>139667</td>
</tr>
<tr>
<td>24 permutations</td>
<td>24 permutations</td>
</tr>
</tbody>
</table>

1 = \textit{AABBBBBB} \quad \text{1 = AABBBBBBB}
2 = \textit{ABABBBBB} \quad \text{2 = ABABBBBBB}
3 = \textit{ABBABBBB} \quad \text{3 = ABBABBBBB}
4 = \textit{ABBBABBB} \quad \text{4 = ABBBABBBB}

Table 3.6: $3 \times 3$ products products containing 2A’s and 6 – 7B’s.
<table>
<thead>
<tr>
<th>3A3B</th>
<th>3A4B</th>
<th>3A5B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordering</td>
<td>Count</td>
<td>Ordering</td>
</tr>
<tr>
<td>123</td>
<td>323986</td>
<td>1234</td>
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<td>132</td>
<td>114744</td>
<td>1243</td>
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<td>213</td>
<td>61405</td>
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<tr>
<td>231</td>
<td>114507</td>
<td>1342</td>
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<tr>
<td>312</td>
<td>61084</td>
<td>1423</td>
</tr>
<tr>
<td>321</td>
<td>324274</td>
<td>1432</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6 permutations</th>
<th>24 permutations</th>
<th>52 permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 = \textit{ABABAB}</td>
<td>1 = \textit{ABABABB}</td>
<td>1 = \textit{ABABBABB}</td>
</tr>
<tr>
<td>2 = \textit{AABABB}</td>
<td>2 = \textit{AABBABB}</td>
<td>2 = \textit{ABAABABB}</td>
</tr>
<tr>
<td>3 = \textit{AAABB}</td>
<td>3 = \textit{AABBB}</td>
<td>3 = \textit{AAABB}</td>
</tr>
<tr>
<td>4 = \textit{AAABB}</td>
<td>4 = \textit{AABBB}</td>
<td>4 = \textit{AAABBB}</td>
</tr>
<tr>
<td>5 = \textit{AAABBB}</td>
<td>5 = \textit{AAABBB}</td>
<td>5 = \textit{AAABBB}</td>
</tr>
</tbody>
</table>

Table 3.7: $2 \times 2$ products containing 3A’s and 3 – 5B’s.
Chapter 4

Forbidden Orders

In this chapter we demonstrate that certain orders are forbidden. For the remainder of this chapter, let $\text{Tr}(B) = x$, $\text{Det}(B) = y$ and $\text{Tr}(ABAB - A^2B^2) = z$.

From Lemma 2.1 we know that $B^2 = xB - yI$. Let’s consider higher powers of such a matrix $B$.

$B^3 = BB^2 = B(xB - yI)$
$= xB^2 - yBI$
$= x(xB - yI) - yBI$
$= (x^2 - y)B - xyI$

In a similar fashion we find:

$B^4 = BB^3 = (x^3 - 2xy)B - (x^2y - y^2)I$
$B^5 = BB^4 = (x^4 - 3x^2y + y^2)B - (x^3y - 2xy^2)I$
$B^6 = BB^5 = (x^5 - 4x^3y + 3xy^3)B - (x^4y - 3x^2y^2 + y^3)I$.

We now introduce two variable Lucas polynomials.

**Definition 4.1.** Define a sequence of polynomials $\{U_n(x, y)\}$ as

$U_0 = 0$,  $U_1 = 1$,  $U_n(x, y) = xU_{n-1}(x, y) - yU_{n-2}(x, y)$

for all $n \geq 2$.
The first several terms of the sequence are given by the following:

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_n(x, y))</td>
<td>0</td>
<td>1</td>
<td>(x)</td>
<td>(x^2 - y)</td>
<td>(x^3 - 2xy)</td>
<td>(x^4 - 3x^2y + y^2)</td>
<td>(x^5 - 4x^3y + 3xy^2)</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 4.1: Two variable Lucas Polynomials

Lucas polynomials, or Lucas sequences, have many uses in linear algebra and number theory. See [2, pp. 393-411], and [8, pp. 41-61], for general discussions of these polynomials.

**Lemma 4.1.** If \(B\) is a \(2 \times 2\) matrix with \(\text{Tr}(B) = x\) and \(\text{Det}(B) = y\), then

\[
B^n = U_n(x, y)B - yU_{n-1}(x, y)I
\]

where \(I\) is the \(2 \times 2\) identity matrix.

**Proof.** We prove this Lemma by induction on \(n\).

Suppose \(n = 1\). Then \(B^1 = 1 \cdot B - 0 \cdot I = U_1B - yU_0I\) as desired.

Suppose \(n = 2\). Then by Lemma 2.1, \(B^2 = xB - yI = U_2B - yU_1I\) as desired.

Inductive Hypothesis: \(B^n = U_n(x, y)B - yU_{n-1}(x, y)I\).

We will show that \(B^{n+1} = U_{n+1}(x, y)B - yU_n(x, y)I\). We have

\[
B^{n+1} = B^nB = [U_n(x, y)B - yU_{n-1}(x, y)I]B \quad \text{by inductive hypothesis}
\]

\[
= U_n(x, y)B^2 - yU_{n-1}(x, y)BI
\]

\[
= U_n(x, y)[xB - yI] - yU_{n-1}(x, y)B \quad \text{by Lemma 2.1}
\]

\[
= xU_n(x, y)B - yU_n(x, y)I - yU_{n-1}(x, y)B
\]

\[
= [xU_n(x, y) - yU_{n-1}(x, y)]B - yU_n(x, y)I
\]

\[
= U_{n+1}(x, y)B - yU_n(x, y)I \quad \text{by Definition 4.1}
\]
Lemma 4.2.

\[ U_{m+n} = xU_m U_n - yU_{m-1} U_{n-1} - yU_{m-1} U_n \]
\[ -yU_{m+n-1} = U_{m-1} U_{n-1} - yU_{m-1} U_n. \]

**Proof.** Using Lemma 4.1

\[ B^m B^n = U_{m+n} B - yU_{m+n-1} I \]
\[ = (U_m B - yU_{m-1} I)(U_n B - yU_{n-1} I) \]
\[ = U_m U_n B^2 - yU_m U_{n-1} B - yU_{m-1} U_n B + y^2 U_{m-1} U_{n-1} I \]
\[ = (xU_m U_n - yU_{m-1} U_{n-1} - yU_{m-1} U_n)B + (y^2 U_{m-1} U_{n-1} - yU_{m-1} U_n)I. \]

This tells us that

\[ U_{m+n} = xU_m U_n - yU_{m-1} U_{n-1} - yU_{m-1} U_n \]
\[ -yU_{m+n-1} = U_{m-1} U_{n-1} - yU_{m-1} U_n. \]

as desired. \(\square\)

We now introduce a formula that relates a polynomial to the difference for the traces of a product of 2 \(\times\) 2 matrices \(A\) and \(B\).

**Theorem 4.3.** If \(0 \leq k \leq m \leq n\) then

\[ Tr(AB^m AB^n - AB^{m-k} AB^{n+k}) = cTr(ABAB - A^2 B^2) \]

where \(c = y^{m-k} U_k U_{n-m+k}. \)

**Proof.** Using Lemma 4.1

\[ B^n B^k = B^{n+k} = U_n B^{k+1} - yU_{n-1} B^k, \]
\[ \Rightarrow B^n = U_{n-k} B^{k+1} - yU_{n-k-1} B^k, \]
\[ \Rightarrow B^n = U_k B^{n-k+1} - yU_{k-1} B^{n-k}. \]
Now

\[ AB^m AB^n = U_m ABAB^n - yU_{m-1}A^2B^n, \]
\[ = U_{m+1}ABAB - yU_mU_{n-2}ABAB \]
\[ - yU_{m-1}U_{n-1}A^2B^2 + y^2U_{m-1}U_{n-2}A^2B. \]

This means that

\[ AB^m AB^n - AB^{m-k}AB^{n+k} = ABAB(U_mU_{n-1} - U_{m-k}U_{n+k-1}) \]
\[ - yABAB(U_mU_{n-2} - U_{m-k}U_{n+k-2}) \]
\[ - yA^2B^2(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) \]
\[ + y^2A^2B(U_{m-1}U_{n-2} - U_{m-k-1}U_{n+k-2}). \]

If we write \( ABAB^2 = xABAB - yABA \) then we have

\[ AB^m AB^n - AB^{m-k}AB^{n+k} = ABAB(xU_mU_{n-1} - xU_{m-k}U_{n+k-1}) \]
\[ - yABAB(U_mU_{n-2} - U_{m-k}U_{n+k-2}) \]
\[ - yA^2B^2(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) \]
\[ - yABA(U_mU_{n-1} - U_{m-k}U_{n+k-1}) \]
\[ + y^2A^2B(U_{m-1}U_{n-2} - U_{m-k-1}U_{n+k-2}). \]

The expression multiplying \( ABAB \)

is \( xU_mU_{n-1} - xU_{m-k}U_{n+k-1} - yU_mU_{n-2} + yU_{m-k}U_{n+k-2} \)
\[ = U_m(xU_{n-1} - yU_{n-2}) - U_{m-k}(xU_{n+k-1} - yU_{n+k-2}) \]
\[ = U_mU_n - U_{m-k}U_{n+k}. \]

This means that

\[ AB^m AB^n - AB^{m-k}AB^{n+k} = ABAB(U_mU_{n-1} - U_{m-k}U_{n+k}) \]
\[ - yA^2B^2(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) \]
\[ - yABA(U_mU_{n-1} - U_{m-k}U_{n+k-1}) \]
\[ + y^2A^2B(U_{m-1}U_{n-2} - U_{m-k-1}U_{n+k-2}). \]
Using Lemma 4.2 we find that \( U_m U_n = U_{m+n-1} + yU_{m-1}U_{n-1} \). If we apply this to the coefficient of \( ABAB \) we have
\[
U_m U_n - U_{m-k} U_{n+k} = U_{m+n-1} + yU_{m-1}U_{n-1} - (U_{m+n-1} + yU_{m-1-k}U_{n-1+k})
\]
\[
= y(U_{m-1}U_{n-1} - U_{m-1-k}U_{n-1+k}).
\]
That is, at the cost of a factor of \( y \), all induces were dropped by 1. If we iterate this \( m-k \) times we get
\[
U_m U_n - U_{m-k} U_{n+k} = y(U_{m-1}U_{n-1} - U_{m-1-k}U_{n-1+k})
\]
\[
= y^2(U_{m-2}U_{n-2} - U_{m-2-k}U_{n-2+k})
\]
\[
= \ldots
\]
\[
= y^{m-k}(U_{m-(m-k)}U_{n-(m-k)} - U_{m-(m-k)-k}U_{n-(m-k)+k})
\]
\[
= y^{m-k}(U_kU_{n-m+k} - U_0 U_{n-m+2k})
\]
\[
= y^{m-k}U_k U_{n-m+k}, \quad \text{since} \quad U_0 = 0.
\]
This means the whole expression simplifies. We have
\[
U_m U_n - U_{m-k} U_{n+k} = y^{m-k}U_k U_{n-m+k},
\]
\[
-y(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) = -y \cdot y^{m-1-k} U_{k} U_{n-m+k},
\]
\[
-y(U_m U_{n-1} - U_{m-k} U_{n+k}) = -y \cdot y^{m-k} U_{k} U_{n-m+k},
\]
\[
y^2(U_{m-1}U_{n-2} - U_{m-1-k}U_{n-2+k}) = y^2 \cdot y^{m-1-k} U_k U_{n-m-1+k}.
\]
This means
\[
AB^m AB^n - AB^{m-k} AB^{n+k} = y^{m-k} U_k U_{n-m+k}(ABAB - A^2 B^2)
\]
\[
- y^{n+1-k} U_k U_{n-m-1+k}(ABA - A^2 B).
\]
Finally, taking the trace and noting that \( \text{Tr}(ABA - A^2 B) = 0 \) we have
\[
\text{Tr}(AB^m AB^n - AB^{m-k} AB^{n+k}) = y^{m-k} U_k U_{n-m+k} \text{Tr}(ABAB - A^2 B^2)
\]
as desired. \( \square \)

As an example, if we let \( z = \text{Tr}(ABAB - A^2 B^2) \), when \( m = n = k = 1 \) we have
\[
\text{Tr}(AB^1 AB^1 - AB^{1-1} AB^{1+1}) = \text{Tr}(ABAB - A^2 B^2) = z.
\]
For \( n = 2 \) we find

\[
Tr(ABAB^2 - A^2B^3) = xz \\
Tr(AB^2AB^2 - ABAB^3) = yz \\
Tr(AB^2AB^2 - A^2B^4) = x^2z
\]

We can use the formula in Theorem 4.3 to derive such tables, then use polynomial inequalities to show certain permutations cannot occur. For example, in Table 3.2 for two A’s and six B’s the ordering 4312 corresponding to \( Tr(A^2B^6) > Tr(ABAB^5) > Tr(AB^3AB^3) > Tr(AB^2AB^4) \) occurred 0 times from 1,000,000 trials. With 1,000,000 trials we expect approximately three decimal places of accuracy, therefore we should be able to show this ordering cannot occur, or that this ordering is forbidden. Using Theorem 4.3

\[
Tr(A^2B^6) > Tr(ABAB^5) > Tr(AB^3AB^3) > Tr(AB^2AB^4)
\]

translates to the set of inequalities

\[
-(x^4 - 3x^2y + y^2)z > 0 \quad (4.1) \\
-x^2yz > 0 \quad (4.2) \\
y^2z > 0 \quad (4.3)
\]

Now (4.3) forces \( z > 0 \), so (4.2) implies that \( y < 0 \). Now (4.1) forces \((x^4 - 3x^2y + y^2)z < 0\), a contradiction.

Many such forbidden orders can be demonstrated in this fashion. Some are trickier. For example, in Table 3.1 the ordering 231 corresponding to \( Tr(ABAB^3) > Tr(A^2B^4) > AB^2AB^2 \) occurred 0 times from 1,000,000 trials. Using Theorem 4.3

\[
Tr(ABAB^3) > Tr(A^2B^4) > AB^2AB^2
\]

translates to

\[
(x^2 - y)z > 0 \quad (4.4) \\
-x^2z > 0. \quad (4.5)
\]

There is no obvious reason that we cannot have \( z < 0 \) and \( x^2 - y < 0 \). To explain these results we need the following:
Theorem 4.4. If $\text{Tr}(ABAB - A^2B^2) < 0$ then $x^2 - 4y \geq 0$.

Proof. If $\text{Tr}(ABAB - A^2B^2) = z < 0$ then $B$ has real eigenvalues, call them $\lambda_1$ and $\lambda_2$.

Then $x^2 - 4y = (\lambda_1 + \lambda_2)^2 - 4(\lambda_1\lambda_2)$ by Lemma 2.2

$= \lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 - 4\lambda_1\lambda_2$

$= \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2$

$= (\lambda_1 - \lambda_2)^2 \geq 0$, as desired.

We can now confirm that $\text{Tr}(ABAB^3) > \text{Tr}(A^2B^4) > AB^2AB^2$ is not possible because (4.4) forces $z < 0$, which implies $x^2 - 4y \geq 0$ by Theorem 4.4. Now this implies $x^2 - y \geq 0$, a contradiction.

Using Theorem 4.3 and Table 4.1 we are able to eliminate cases of trace inequalities to more efficiently verify which orders are forbidden. We use the same notation found in Table 3.2 for example $1 > 2$ if $\text{Tr}(ABAB^3) > \text{Tr}(AB^2AB^4)$. The associated polynomial with this order is obtained using Theorem 4.3. For $1 > 2$ we have $m = 3, n = 3$ and $k = 1$, so the resulting polynomial inequality will be $y^{3-1}U_1U_3-3+1\text{Tr}(ABAB - A^2B^2) > 0$ so $y^2U_1U_1z > 0$, or $y^2z > 0$.

For $m + n = 6$ we have:

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Order & 1 > 2 & 1 > 3 & 1 > 4 & 2 > 3 & 2 > 4 & 3 > 4 \\
\hline
Polynomial & $y^2U_1^2z$ & $y^1U_2^2z$ & $y^0U_3^2z$ & $y^1U_1U_3z$ & $y^0U_2U_4z$ & $y^0U_1U_5z$ \\
\hline
\end{tabular}
\caption{2A6B Polynomials}
\end{table}

or,

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Order & 1 > 2 & 1 > 3 & 1 > 4 & 2 > 3 & 2 > 4 & 3 > 4 \\
\hline
Polynomial & $y^2z$ & $yx^2z$ & $(x^2 - y)^2z$ & $y(x^2 - y)z$ & $x^2(x^2 - 2y)z$ & $(x^4 - 3x^2y + y^2)z$ \\
\hline
\end{tabular}
\caption{2A6B Trace Inequalities}
\end{table}
For two $A$’s and six $B$’s we can obtain all possible polynomials from Table 4.3. The order $2 > 1$ would correspond to $\text{Tr}(AB^3AB^3) > \text{Tr}(AB^2AB^4)$, or $-1(\text{Tr}(AB^3AB^3 - AB^2AB^4)) > 0$ or $-y^2z > 0$. That is, 1 is to the left of 2 in any allowable permutation when $y^2z$ is positive and 1 is to the right of 2 when $y^2z$ is negative. In general, if there are an even number of $B$’s, say $2n$ total, then the product will have the form $y^jU_kU_{2n+k}z$ for some $j$ and $k$.

**Theorem 4.5.** If $n - m$ is even, $y > 0$ and $z < 0$ then $U_kU_{n-m+k} > 0$.

**Proof.** Since $z < 0$ we know that $x^2 - 4y > 0$ by Theorem 4.4. By formula (IV.8) in [8] we have the formula

$$2^{n-1}U_n = \left(\begin{array}{c} n \\ 1 \end{array}\right)x^{n-1} + \left(\begin{array}{c} n \\ 3 \end{array}\right)x^{n-3}D + \left(\begin{array}{c} n \\ 5 \end{array}\right)x^{n-5}D^2 + ...$$

where $D = x^2 - 4y$. This means that

$$2^{k-1}U_k2^{n-m+k-1}U_{n-m+k}$$

$$= ((\begin{array}{c} k \\ 1 \end{array})x^{k-1} + (\begin{array}{c} k \\ 3 \end{array})x^{k-3}D + (\begin{array}{c} k \\ 5 \end{array})x^{k-5}D^2 + ...)((\begin{array}{c} j \\ 1 \end{array})x^{j-1} + (\begin{array}{c} j \\ 3 \end{array})x^{j-3}D + (\begin{array}{c} j \\ 5 \end{array})x^{j-5}D^2 + ...)$$

$$= (\begin{array}{c} k \\ 1 \end{array})(\begin{array}{c} j \\ 1 \end{array})x^{j+k-2} + ((\begin{array}{c} k \\ 3 \end{array})(\begin{array}{c} j \\ 3 \end{array}) + (\begin{array}{c} k \\ 5 \end{array})(\begin{array}{c} j \\ 5 \end{array}))x^{j+k-4}(x^2 - 4y) + ((\begin{array}{c} k \\ 1 \end{array})(\begin{array}{c} j \\ 5 \end{array}) + (\begin{array}{c} k \\ 3 \end{array})(\begin{array}{c} j \\ 3 \end{array}) + (\begin{array}{c} k \\ 5 \end{array})(\begin{array}{c} j \\ 1 \end{array}))x^{j+k-6}(x^2 - 4y)^2 + ...$$

where $j = n - m + k$. This means that $j + k = n - m + 2k$ is even, so all powers of $x$ are even. Thus, we have collections of binomial coefficients, which are positive, times even powers of $x$, which are positive, times powers of $x^2 - 4y$, which are positive. That is, every term in the product of $U_kU_{n-m+k}$ is positive, meaning that the product itself must be positive. \qed

Now, if $y < 0$, Table 4.3 becomes:

<table>
<thead>
<tr>
<th>Order</th>
<th>1 &gt; 2</th>
<th>1 &gt; 3</th>
<th>1 &gt; 4</th>
<th>2 &gt; 3</th>
<th>2 &gt; 4</th>
<th>3 &gt; 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial</td>
<td>$z$</td>
<td>$-z$</td>
<td>$z$</td>
<td>$-z$</td>
<td>$z$</td>
<td>$z$</td>
</tr>
</tbody>
</table>

Table 4.4: 2A6B Trace Inequalities for $y < 0$
This leads to two possible sign patterns, depending on when $z$ is positive or $z$ is negative. When $z$ is positive, the only permutation possible is 3124, similarly when $z$ is negative the only permutation possible is 4213.

Now if $y > 0$ and $z < 0$ then by Theorem 4.5, Table 4.3 has the form:

<table>
<thead>
<tr>
<th>Order</th>
<th>1 &gt; 2</th>
<th>1 &gt; 3</th>
<th>1 &gt; 4</th>
<th>2 &gt; 3</th>
<th>2 &gt; 4</th>
<th>3 &gt; 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial</td>
<td>(−)</td>
<td>(−)</td>
<td>(−)</td>
<td>(−)</td>
<td>(−)</td>
<td>(−)</td>
</tr>
</tbody>
</table>

Table 4.5: 2A6B Trace Inequalities for $y > 0$ and $z < 0$

This leads to only one possible permutation, namely 4321.

Finally, if $y > 0$ and $z > 0$, Table 4.3 becomes:

<table>
<thead>
<tr>
<th>Order</th>
<th>1 &gt; 2</th>
<th>1 &gt; 3</th>
<th>1 &gt; 4</th>
<th>2 &gt; 3</th>
<th>2 &gt; 4</th>
<th>3 &gt; 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial</td>
<td>(+)</td>
<td>(+)</td>
<td>(+)</td>
<td>$x^2 - y$</td>
<td>$x^2 - 2y$</td>
<td>$x^4 - 3x^2y + y^2$</td>
</tr>
</tbody>
</table>

Table 4.6: 2A6B Trace Inequalities for $y > 0$ and $z > 0$

We quickly see the only possible orderings remaining will begin with 1. Now everything depends on the possible signs of the polynomials involved. These signs depend on how $x^2$ compares to $y$. We can scale the $y$ away by setting it equal to 1, and the problem reduces to how the zeros of $U_n(x, 1)$ are arranged. We also replace $x^2$ by $x$ so the degrees drop by a factor of 2 while everything else remains unchanged. Thus we need to know about the zeros of $x - 1, x - 2, \text{ and } x^2 - 3x + 1$. Note that the zeros of $x^2 - 3x + 1$ are approximately 0.382 and 2.618. Now, if we take all these zeros and put them in numerical order, then every time $x$ passes from one range to another, some collection of signs will change. It is not hard to check that all sign patterns are different, as shown in Table 4.7.
Table 4.7: Sign Patterns

<table>
<thead>
<tr>
<th></th>
<th>$x \in (0, 0.382)$</th>
<th>$x \in (0.382, 1)$</th>
<th>$x \in (1, 2)$</th>
<th>$x \in (2, 2.618)$</th>
<th>$x = (2.618, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x - 1$</td>
<td>(-)</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>$x - 2$</td>
<td>(-)</td>
<td>(-)</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>$x^2 - 3x + 1$</td>
<td>(+)</td>
<td>(-)</td>
<td>(-)</td>
<td>(-)</td>
<td>(+)</td>
</tr>
</tbody>
</table>

Since we have verified that all sign patterns are different, we find the number of possible permutations is one more than the total number of distinct zeros among the $U_n$. Counting these zeros we get $1 + 1 + 2 = 4$ distinct zeros, and thus 5 total permutations.

Putting everything together, we have shown that of the 24 possible orderings when considering a product of two $A$’s and six $B$’s, only 8 permutations are actually possible; three which do not begin with 1 and five which do being with 1. Note, we are not saying these permutations must occur, but rather that these are the only permutations which could possible occur. Our data in Table 3.2 verifies these orders do indeed occur, so we have verified there are 16 forbidden orders.

Table 4.8 summarizes the number of occurring orders vs the number of possible orders for our simulations.
<table>
<thead>
<tr>
<th>Number of B’s</th>
<th>Necklaces</th>
<th>Possible Orders</th>
<th>Orders Occurring</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>24</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>120</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>120</td>
<td>22</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>720</td>
<td>17</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>720</td>
<td>32</td>
</tr>
<tr>
<td>12</td>
<td>7</td>
<td>5040</td>
<td>24</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>5040</td>
<td>46</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>40320</td>
<td>32</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>40320</td>
<td>64</td>
</tr>
</tbody>
</table>

Table 4.8: Confirmed Simulations

This same process can be extended to other combinations with more B’s. The advantage of this process is the ability to eliminate large numbers of possibilities at once. For example, with two A’s and ten B’s there are 720 possible orderings, of which only 17 occurred. It would be a tedious exercise to verify each of the 703 forbidden orders one by one. We construct Table 4.10 to demonstrate. In this table, we write 12 rather than 1 > 2. The meaning is the same as before, 1 is to the left of 2 if and only if $\gamma^4z$ is positive.
If $y < 0$, Table 4.10 reduces to:

<table>
<thead>
<tr>
<th>Order</th>
<th>Polynomial</th>
<th>Order</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$z$</td>
<td>26</td>
<td>$z$</td>
</tr>
<tr>
<td>13</td>
<td>$-z$</td>
<td>34</td>
<td>$z$</td>
</tr>
<tr>
<td>14</td>
<td>$z$</td>
<td>35</td>
<td>$-z$</td>
</tr>
<tr>
<td>15</td>
<td>$-z$</td>
<td>36</td>
<td>$z$</td>
</tr>
<tr>
<td>16</td>
<td>$z$</td>
<td>45</td>
<td>$-z$</td>
</tr>
<tr>
<td>23</td>
<td>$-z$</td>
<td>46</td>
<td>$z$</td>
</tr>
<tr>
<td>24</td>
<td>$z$</td>
<td>56</td>
<td>$z$</td>
</tr>
<tr>
<td>25</td>
<td>$-z$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.10: 2A10B Trace Inequalities for $y < 0$

This leads to two possible sign patterns, depending on when $z$ is positive or $z$ is negative. When $z$ is positive, the only permutation possible is 531246, similarly when $z$ is negative the only permutation possible is 642135. Next, much like the 2A6B case, when ($y > 0$ and $z < 0$), all polynomials are negative by Theorem 4.5, resulting in one possible permutation, 654321. In general, when considering both $y < 0$ or ($y > 0$ and $z < 0$), if the number of $B$’s is even, there will be a total of 3 possible permutations, all of which do not begin with 1. All other allowable permutations are determined by combinations of the signs of the polynomials when ($y > 0$ and $z > 0$). This leads us to
the following conjectures.

**Conjecture 4.1.** For a product of $2 \times 2$ matrices containing two $A$’s with $nB$’s, where $n$ is even, if $y > 0$ and $z > 0$, there are $1 + k$ possible permutations, where $k$ is the number of distinct positive zeros of $U_m(x, 1)$, for $3 \leq m \leq n – 1$. Consequently, the total number of possible permutations is $4 + k$.

For the case above, $n = 10$, the number of distinct zeros is 13, therefore we have $4 + 13 = 17$ possible permutations as observed in Table 4.8.

When there is an odd number $n$ of $B$’s, the permutations which arise from both $y < 0$ or $(y > 0$ and $z < 0)$ are repeated in the situation with $(y > 0$ and $z > 0)$. So it is enough to only consider this case.

**Conjecture 4.2.** For a product of $2 \times 2$ matrices containing two $A$’s with $nB$’s, where $n$ is odd, there are $2(1 + k)$ possible permutations, where $k$ is the number of distinct positive zeros of $U_m(x, 1)$, for $3 \leq m \leq n – 1$.

As an example, when considering the case of two $A$’s seven $B$’s, it can be shown using Theorem 4.3 that there are 5 distinct positive zeros. Therefore we expect $2(1 + 5) = 12$ possible permutations, as observed in Table 4.8.
Chapter 5

Future Work

There are several ways this work might be extended. To begin with, there is a simple extension to larger matrices. Data collected on products consisting of larger matrices suggests that forbidden orders are not limited to only the $2 \times 2$ case. We collected some data on products of two $A$’s with $nB$’s for $3 \times 3$ and $4 \times 4$ matrices $A$ and $B$. For relatively small values of $n$, there appeared to be no forbidden orders. However, as $n$ increases data suggests that orderings are a best, very unlikely, which leads us to believe that forbidden orders are prevalent for larger and larger $n$. In the $3 \times 3$ case, it was not until the number of $B$’s reached seven until zeros start appearing in our simulations.

Another extension would include products of matrices with more $A$’s and $B$’s. For a product of $mA$’s with $nB$’s, we only looked at the case of $m = 3$ and $n = 3, 4, 5$ for $2 \times 2$ matrices $A$ and $B$, forbidden orders started to appear when $n = 5$. Extending this to larger matrices and increasing both $m$ and $n$ could yield interesting results. One could also consider the effect that eigenvalues are having on products with more $A$’s and $B$’s.

Finally, it would be worth looking into proving the conjectures listed in Chapter 4. With these one could potentially derive a formula for the number of forbidden orders when taking a product of $mA$’s and $nB$’s. Or in another direction, deriving a formula similar to Theorem 4.3 but extending this to products with more than two $A$’s would be extremely useful.
Chapter 6

Appendix. Source Code

In this section we provide several examples of the code used for our simulations. All code presented is written in Mathematica, and a brief description of what each piece of code does is presented.
Here is a sample of code used to collect data on nxn matrices, as n varies from 2 to 1000. This code takes advantage of parallel computing to increase the speed. The main idea behind this code is to have several different workstations collecting and sending data into one global list.

```mathematica
globalTable = Table[0, {i, 1, 2}];
set[] := Module[{},
    localTable = Table[0, {i, 1, 2}];
];
unset[] := Module[{},
    Clear[localTable];
];
do[] := Module[{{a, b, c, d},
    a = RandomReal[NormalDistribution[0, 1], {1000, 1000}];
    b = RandomReal[NormalDistribution[0, 1], {1000, 1000}];
    c = SetPrecision[Tr[Dot[a, a, b]], 100];
    d = SetPrecision[Tr[Dot[a, b, a]], 100];
    If[c > d, localTable[[1]]++, localTable[[2]]++]
];
gether[] := Module[{},
    globalTable += localTable;
];
DistributeDefinitions[set, unset, do, gether];
SetSharedVariable[globalTable];
globalTable = Table[0, {i, 1, 2}];
ParallelEvaluate[set[]];
AbsoluteTiming[
    ParallelDo[
        Monitor[do[], {i, 1, 100}];
        ParallelEvaluate[gether[]];
        ParallelEvaluate[unset[]];
    ]
] (* 25.728.1080503, Null *)
globalTable is where all the individual simulations were collected, 281519 represents the number of occurrences that \( \text{Tr}(AAB-B) > \text{Tr}(ABAB) \), and 718481 represents the number of occurrences that \( \text{Tr}(ABAB) > \text{Tr}(AABB) \).

```mathematica

globalTable

{281519, 718481}

Here I am verifying that the globalTable sums to 1,000,000.

```mathematica
Total[globalTable]

1 000 000
```
This is a sample of code used to collect eigenvalue data for 2x2 matrices A and B. g and h represent the discriminant of the characteristic polynomial of matrix A or B, respectively. The given matrix has complex eigenvalues if this discriminant is negative.

```mathematica
f = Table[0, {i, 1, 8}];

AbsoluteTiming[
  For[i = 10^6, i > 0, i--,
    a = RandomReal[NormalDistribution[0, 1], {2, 2}];
    b = RandomReal[NormalDistribution[0, 1], {2, 2}];
    
    c = SetPrecision[Tr[Dot[a, a, b, b]], 100];
    d = SetPrecision[Tr[Dot[a, a, b, b]], 100];
    
    g = (a[[1, 1]])^2 + 4 (a[[1, 2]]) (a[[2, 1]]) - 2 (a[[1, 1]]) (a[[2, 2]]) + (a[[2, 2]])^2;
    h = (b[[1, 1]])^2 + 4 (b[[1, 2]]) (b[[2, 1]]) - 2 (b[[1, 1]]) (b[[2, 2]]) + (b[[2, 2]])^2;

    (* real real *)
    If[g > 0 && h > 0,
      If[c > d, f[[1]] ++, (*12*)
        If[d > c, f[[2]] ++]], (*21*)
      (* real complex *)
      If[g > 0 && h < 0,
        If[c > d, f[[3]] ++, (*12*)
          If[d > c, f[[4]] ++]], (*21*)
        (* complex real *)
        If[g < 0 && h > 0,
          If[c > d, f[[5]] ++, (*12*)
            If[d > c, f[[6]] ++]], (*21*)
          (* complex complex *)
          If[g < 0 && h < 0,
            If[c > d, f[[7]] ++, (*12*)
              If[d > c, f[[8]] ++]], (*21*)
            ]
          ]
    ]
  ]

{106.7074112, Null}

f

{250 007, 249 826, 103 847, 103 658, 103 445, 103 554, 42 791, 42 872}

Sum[f[[i]], {i, 1, 8}]

1 000 000
This is a sample of code used to collect eigenvalue data for 4x4 matrices A and B. Here we also take advantage of parallel computing to increase speed. To determine if a 4x4 matrix A has 0, 2, or 4 real eigenvalues we perform the following algorithm. If the Discriminant of the Characteristic Polynomial < 0 and the Minimum Value of this Discriminant > 0 then 4 real eigenvalues, if the Discriminant of the Characteristic Polynomial < 0 and the Minimum Value of this Discriminant < 0 then 2 real eigenvalues and if the Discriminant of the Characteristic Polynomial > 0 and the Minimum Value of this Discriminant > 0 then 0 real eigenvalues.

```
globalTable = Table[0, {i, 1, 18}];
set[] := Module[{},
   localTable = Table[0, {i, 1, 18}];
];
unset[] := Module[{},
   Clear[localTable];
];
do[] := Module[{a, b, c, d, e, f, g, h, i, j},
   a = RandomReal[NormalDistribution[0, 1], {4, 4}];
   b = RandomReal[NormalDistribution[0, 1], {4, 4}];

   c = CharacteristicPolynomial[a, x];
   d = CharacteristicPolynomial[b, x];

   e = Discriminant[c, x];
   f = Discriminant[d, x];

   g = NMinValue[c, x, MaxIterations -> 10];
   (* Global Min for Matrix a Characteristic Polynomial *)
   h = NMinValue[d, x, MaxIterations -> 10];
   (* Global Min for Matrix b Characteristic Polynomial *)

   i = SetPrecision[Tr[Dot[a, a, b, b]], 20];
   j = SetPrecision[Tr[Dot[a, b, a, b]], 20];

   If[e > 0 && g < 0,
    If[f > 0 && h < 0 && i > j, localTable[[1]]++,
     If[f > 0 && h < 0 && i < j, localTable[[2]]++,
      If[f < 0 && i > j, localTable[[3]]++,
       If[f < 0 && i < j, localTable[[4]]++,
        If[f > 0 && h > 0 && i > j, localTable[[5]]++,
         If[f > 0 && h > 0 && i < j, localTable[[6]]++
          ]]]]]];

   If[e < 0 && f > 0,
    If[h < 0 && i > j, localTable[[7]]++,
     If[h < 0 && i < j, localTable[[8]]++,
      If[h > 0 && i > j, localTable[[11]]++,
       If[h > 0 && i < j, localTable[[12]]++
        ]]]]]];
```
If \( e < 0 \land f < 0 \),
If \( i > j \), localTable[[9]]++;
If \( i < j \), localTable[[10]]++;]

If \( e > 0 \land g > 0 \),
If \( f > 0 \land h < 0 \land i > j \), localTable[[13]]++;
If \( f > 0 \land h < 0 \land i < j \), localTable[[14]]++;
If \( f < 0 \land i > j \), localTable[[15]]++;
If \( f < 0 \land i < j \), localTable[[16]]++;
If \( f > 0 \land h > 0 \land i > j \), localTable[[17]]++;
If \( f > 0 \land h > 0 \land i < j \), localTable[[18]]++;]

};
gather[] := Module[{},
  globalTable += localTable;
];

DistributeDefinitions[set, unset, do, gather];
SetSharedVariable[globalTable];
globalTable = Table[0, {i, 1, 18}];
ParallelEvaluate[set[]];
AbsoluteTiming[
  ParallelDo[
    do[], {i, 1, 10^6}];
  ParallelEvaluate[gather[]];
  ParallelEvaluate[unset[]];
]
{2338.3597465, Null}

globalTable
{10 393, 5271, 34 885, 55 621, 4845, 14 395, 34 591, 55 706, 151 412, 369 583, 26 145, 84 145, 4650, 14 250, 26 065, 84 794, 5410, 17 839}

Total@globalTable
1 000 000
This is a sample of code used to collect trace data for 2x2 matrices $A$ and $B$. Here we create a list of all the possible permutations of the string abcd. We then compute the values for $a,b,c,d$ and sort them. Once sorted, we increase the count in our table corresponding to that sorted permutation.

$$z = \text{Table}[0, \{i, 1, 24\}]$$
$$p = \text{List["c", "d", "e", "f"};$$
$$\text{perm} = \text{Permutations}[p];$$
$$\text{AbsoluteTiming[}$$
$$\text{For}[i = 10^6, i > 0, i--,$$
$$a = \text{SetPrecision[RandomReal[NormalDistribution[0, 1], \{2, 2\}], 100];}$$
$$b = \text{SetPrecision[RandomReal[NormalDistribution[0, 1], \{2, 2\}], 100];}$$
$$c = \text{SetPrecision[Tr[Dot[a, b, b, a, b, b, b]], 100];}$$
$$d = \text{SetPrecision[Tr[Dot[a, b, b, a, b, b, b]], 100];}$$
$$e = \text{SetPrecision[Tr[Dot[a, b, a, b, b, b]], 100];}$$
$$f = \text{SetPrecision[Tr[Dot[a, a, b, b, b, b]], 100];}$$
$$\text{temp} = \{\{c, "c"\}, \{d, "d"\}, \{e, "e"\}, \{f, "f"\};$$
$$\text{temp} = \text{Sort[temps];}$$
$$z[[\text{Flatten@Position[perm, temp[[\{All, 2\}]}[[1]]]]]]++;$$
$$\}$$
$$\{445.0324544, \text{Null}\}$$

Here we print the counts for each permutation of abcd. Note: The permutations are listed in the reverse order, that is, \{c,d,e,f\} is actually \{f,e,d,c\}, something we account for when inputing the data to a table.

$$\text{For}[i = 0, i < 24, i++$$
$$\text{If}[z[[i]] > 0, \text{Print[z[[i]], perm[[i]]]]]$$

75316\{c, d, e, f\}
48795\{d, e, f, c\}
75127\{d, f, e, c\}
217089\{e, c, d, f\}
57683\{e, d, f, c\}
32183\{e, f, d, c\}
282590\{f, d, c, e\}
211217\{f, e, d, c\}

This is what all the counts look like, including those which do not occur.

$$z$$

\{75316, 0, 0, 0, 0, 0, 0, 0, 48795, 0, 75127,
217089, 0, 0, 57683, 0, 32183, 0, 0, 282590, 0, 0, 211217\}
This is a sample of code used to collect trace data for 2x2 matrices A and B. Here we are verifying that the Lists code on the previous page is working as intended. Each permutation of the string abcd is manually checked in nested If statements. There is no speed advantage to the List code, however there is a huge advantage in that we do not have to write out 8! = 40,320 nested If statements when in the 2A2B case for example.

\[\begin{aligned}
\text{In}[1]:&= z = \text{Table}[0, \{i, 1, 24\}]; \\
\text{In}[2]:&= \text{AbsoluteTiming[For[i = 10^6, i > 0, i--,}
\begin{align*}
a &= \text{SetPrecision[RandomReal[NormalDistribution[0, 1], \{2, 2\}], 100];} \\
b &= \text{SetPrecision[RandomReal[NormalDistribution[0, 1], \{2, 2\}], 100];} \\
c &= \text{SetPrecision[Tr[Dot[a, b, b, a, b, b]], 100];} \\
d &= \text{SetPrecision[Tr[Dot[a, b, b, a, b, b]], 100];} \\
e &= \text{SetPrecision[Tr[Dot[a, b, b, b, b]], 100];} \\
f &= \text{SetPrecision[Tr[Dot[a, b, b, b, b]], 100];} \\
\text{If[c > d > e > f, z[[1]] ++, (*1234*)]}
\end{align*}
\begin{align*}
&\text{If[c > d > f > e, z[[2]] ++, (*1243*)]}
\end{align*}
\begin{align*}
&\text{If[c > e > d > f, z[[3]] ++, (*1324*)]}
\end{align*}
\begin{align*}
&\text{If[c > e > f > d, z[[4]] ++, (*1342*)]}
\end{align*}
\begin{align*}
&\text{If[c > f > d > e, z[[5]] ++, (*1423*)]}
\end{align*}
\begin{align*}
&\text{If[c > f > e > d, z[[6]] ++, (*1432*)]}
\end{align*}
\begin{align*}
&\text{If[d > c > e > f, z[[7]] ++, (*2134*)]}
\end{align*}
\begin{align*}
&\text{If[d > c > f > e, z[[8]] ++, (*2143*)]}
\end{align*}
\begin{align*}
&\text{If[d > e > c > f, z[[9]] ++, (*2314*)]}
\end{align*}
\begin{align*}
&\text{If[d > e > f > c, z[[10]] ++, (*2341*)]}
\end{align*}
\begin{align*}
&\text{If[d > f > c > e, z[[11]] ++, (*2413*)]}
\end{align*}
\begin{align*}
&\text{If[d > f > e > c, z[[12]] ++, (*2431*)]}
\end{align*}
\begin{align*}
&\text{If[e > c > d > f, z[[13]] ++, (*3124*)]}
\end{align*}
\begin{align*}
&\text{If[e > c > f > d, z[[14]] ++, (*3142*)]}
\end{align*}
\begin{align*}
&\text{If[e > d > c > f, z[[15]] ++, (*3214*)]}
\end{align*}
\begin{align*}
&\text{If[e > d > f > c, z[[16]] ++, (*3241*)]}
\end{align*}
\begin{align*}
&\text{If[e > f > c > d, z[[17]] ++, (*3412*)]}
\end{align*}
\begin{align*}
&\text{If[e > f > d > c, z[[18]] ++, (*3421*)]}
\end{align*}
\begin{align*}
&\text{If[f > c > d > e, z[[19]] ++, (*4123*)]}
\end{align*}
\begin{align*}
&\text{If[f > c > e > d, z[[20]] ++, (*4132*)]}
\end{align*}
\begin{align*}
&\text{If[f > d > c > e, z[[21]] ++, (*4213*)]}
\end{align*}
\begin{align*}
&\text{If[f > d > e > c, z[[22]] ++, (*4231*)]}
\end{align*}
\begin{align*}
&\text{If[f > e > c > d, z[[23]] ++, (*4312*)]}
\end{align*}
\begin{align*}
&\text{If[f > e > d > c, z[[24]] ++ (*4321*)]}
\end{align*}
\begin{align*}
&\text{]}
\end{align*}
\begin{align*}
\text{Out}[2]:&= \{406.9942788, \text{Null}\}
\end{align*}
\end{aligned}
In[3]:= \textbf{z}

Out[3]= \{210\,673,\,31\,829,\,0,\,75\,256,\,57\,911,\,48\,940,\,0,\,0,\,0,\,0,\,282\,720,\,0,\,0,\,0,\,0,\,0,\,217\,099,\,0,\,0,\,75\,572\}\n
In[5]:= \textbf{Sum[z[[i]],\{i,1,24\}]} 

Out[5]= 1\,000\,000
References


