Properly Even Harmonious Graphs

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There are many people that have earned my gratitude for their contribution to my time in graduate school.
Dedication

To those who held me up over the years.
Abstract

A function \( f \) is said to be an even harmonious labeling of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the integers from 0 to \( 2q \) and the induced function \( f \) from the edges of \( G \) to \{0, 2, \ldots, 2q - 2\} defined by \( f(uv) = f(u) + f(v) \mod 2q \) is bijective.

A graph \( G \) with \( q \) edges is said to be properly even harmonious if the vertices of \( G \) can be labeled with the integers from 0 to \( 2q - 1 \) without repetition such that when the edge with vertex labels \( x \) and \( y \) is given the label \( x + y \mod 2q \), the edge labels are \{0, 2, \ldots, 2q - 2\}.

In this thesis, we focus on the existence of properly even harmonious labelings for the union of paths, the union of paths and stars, and the union of paths and cycles.
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1 Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. A vertex labeling of a graph $G$ is a mapping $f$ from the vertices of $G$ to a set of elements that induces for each edge $xy$ a label depending on the vertex labels $f(x)$ and $f(y)$. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years nearly 200 graph labelings techniques have been studied in over 2000 papers. In 1980, Graham and Sloane [6] introduced the idea of harmonious labelings in connection with error-correcting codes and channel assignments problems.

2 Preliminaries

In this section we provide definitions and related background to harmonious graph labeling.

**Definition 2.1.** A graph $G$ with $q$ edges is said to be harmonious if there exists an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $f(x) + f(y)(\text{mod } q)$, the resulting edge labels are distinct.

Among the graphs known to be harmonious are wheels, grids $P_m \times P_n$ except when $(m, n) = (2, 2)$, odd cycles, $C_n \times P_{2n+1}$, the $n$-cube $P_2 \times P_2 \times \cdots \times P_2$ for $n \geq 4$, $K_4 \times P_n$, and $nG$, the disjoint of any odd number $n$ of copies a harmonious graph $G$ (see [1]). Figure 1 shows a harmonious labeling for $C_7$.

![Figure 1: Harmonious labeling of $C_7$ (mod 7).](image)

There are many variations of harmonious labelings.

**Definition 2.2.** A function $f$ is said to be an odd harmonious labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q - 1$ and the induced function $f^*$ from the edges of $G$ to 1, 3, \ldots, $2q - 1$ defined by $f^*(xy) = f(x) + f(y)(\text{mod } 2q)$ is bijective. A graph that has an odd harmonious labeling is called an odd harmonious graph.

Among the graphs known to be odd harmonious are $P_n$ ($n \geq 1$), $C_n$ ($n = 0 \text{ mod } 4$), $C_n \times P_m$ ($n = 0 \text{ mod } 4$), $P_m \times P_n$, and the complete bipartite graph
Figure 2: Odd harmonious labeling of $C_8$ (mod 16).

**Definition 2.3.** A function $f$ is said to be an *even harmonious labeling* of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q$ and the induced function $f^*$ from the edges of $G$ to $0, 2, \ldots, 2q - 2$ defined by $f^*(xy) = f(x) + f(y) \mod 2q$ is bijective. A graph that has an even harmonious labeling is called an *even harmonious graph*.

Figure 3 shows an example of an even harmonious labeling. Other examples of even harmonious labeling include: $S_m \cup P_n, C_n \cup P_3$ (*n* odd) and $C_4 \cup P_n$.

Figure 3: Even harmonious labeling of $5P_2$ (mod 10).

**Definition 2.4.** A graph $G$ with $q$ edges is said to be *properly even harmonious* if the vertices of $G$ can be labeled with the integers from 0 to $2q - 1$ without repetition such that when the edge with vertex labels $x$ and $y$ is given the label $x + y \mod 2q$, the edge labels are $\{0, 2, \ldots, 2q - 2\}$. A graph that has a properly even harmonious labeling is called a *properly even harmonious graph*.

Figure 4 shows an example of a properly even harmonious labeling. Notice that the labeling in Figure 3 is not properly even harmonious.

Because any harmonious labeling of a connected graph can be modified to yield an even harmonious labeling, Gallian and Schoenhard [2] investigated even harmonious labelings of disconnected graphs.
Figure 4: Properly even harmonious labeling of $P_7 \cup S_5 \pmod{22}$.

3Disconnected graphs

Gallian and Stewart [3] proved that $nP_m$ is properly even harmonious for $n$ even and $m \geq 2$. Our first result concerns $nP_3$.

**Theorem 3.1.** The graph $nP_3$, where $n > 1$ is odd, is properly even harmonious.

**Proof.** The number of edges of the graph $nP_3$ is $q = 2n$. The modulus is $2q = 4n$.

The cases $n = 3, 5$ are shown in Figures 5 and 6.

Figure 5: Properly even harmonious labeling of $3P_3 \pmod{12}$.

Figure 6: Properly even harmonious labeling of $5P_3 \pmod{20}$. 


Assume $n \geq 7$. See Figures 7 and 8.

**Step 1**
Label the first vertex of each of the first $n - 1$ paths as $0, 1, \ldots, n - 2$.

**Step 2**
Label the second vertex of each of the first $n - 1$ paths as $n - 1, n, \ldots, 2n - 3$.

**Step 3**
Label the last vertex of each of the first $n - 1$ paths as $2n + 2, 2n + 3, \ldots, 3n$.

**Step 4**
Label the first vertex of the last path as $\frac{7n - 1}{2}$, the second one as $\frac{7n - 5}{2}$, and finally the last vertex as $\frac{7n + 3}{2}$.

\[
\begin{align*}
0 & \quad 6 \quad 16 \\
1 & \quad 7 \quad 17 \\
2 & \quad 10 \quad 18 \\
3 & \quad 9 \quad 19 \\
4 & \quad 10 \quad 20 \\
5 & \quad 11 \quad 21 \\
24 & \quad 18 \quad 26
\end{align*}
\]

Figure 7: Properly even harmonious labeling of $7P_3 \pmod{28}$.

Since the vertex labels of the first $n - 1$ paths are strictly increasing and less than the three distinct labels on the last path, which are less than the modulus, there is no duplication of the vertex labels. Likewise, the edge labels are strictly increasing until they wrap around the modulus and increase until they end at 2 less than the initial edge label. So they are distinct.
We use $S_k$ to denote the complete bipartite graph $K_{1,k}$. Gallian and Schoenhard [2] gave a harmonious labelings for $P_n \cup S_m$ and $P_n \cup S_{m_1} \cup S_{m_2}$. In [3, Thm 3.11], Gallian and Stewart proved that $2P_m \cup 2P_n$ is properly even harmonious for $m, n > 1$. Our next results add an extra component.

**Theorem 3.2.** The graph $2P_m \cup 2P_n \cup S_k$ is properly even harmonious for $m, n > 1$.

**Proof.** The number of edges of the graph $2P_m \cup 2P_n \cup S_k$ is $q = 2m + 2n + k - 4$. The modulus is $2q = 4m + 4n + 2k - 8$. See Figures 9 and 10. We may assume that $m \geq n$.

**Step 1**
Label the first vertex of the first path of length $m$ with $4m - 6$, and label the remaining vertices of the first path of length $m$ with $0, 2, \ldots, 2m - 4$. The corresponding edge labels are $4m - 6, 2, 6, 10, \ldots, 4m - 10$.
Label the first vertex of the second path of length $m$ with $4m - 5$, and label its remaining vertices with $1, 3, \ldots, 2m - 3$. The corresponding edge labels are $4m - 4, 4, 8, 12, \ldots, 4m - 8$.

**Step 2**
Label the first $n - 1$ vertices of the first path of length $n$ with $2m - 2, 2m, \ldots, 2m + 2n - 6$, and label its last vertex with $2q - 2m - 2n + 6$. The corresponding edge labels are $4m - 2, 4m + 2, \ldots, 4m + 4n - 14, 0$.
Label the first \( n - 1 \) vertices of the second path of length \( n \) with \( 2m - 1, 2m + 1, \ldots, 2m + 2n - 5 \), and label the last vertex with \( 2q - 2m - 2n + 3 \). The corresponding edge labels are \( 4m, 4m + 4, \ldots, 4m + 4n - 12 \).

**Step 3**
Label the hub of the star as \( 2q - 1 \). Label the leaves as \( 2q - 3, 2q - 5, \ldots, 2q - 2k - 1 \).

To show there is no duplication of the edge and vertex labels note that the vertex labels and edge labels of the paths are strictly increasing and are at most the modulus, so they are distinct.

---

![Figure 9: Properly even harmonious labeling of \( 2P_5 \cup 2P_4 \cup S_5 \mod 38 \).](image)

![Figure 10: Properly even harmonious labeling of \( 2P_7 \cup 2P_4 \cup S_6 \mod 48 \).](image)
Figure 11: Properly even harmonious labeling of $2P_6 \cup 2P_3 \cup S_5 \pmod{38}$.

For Theorems 3.3 through Theorem 3.7 our labeling algorithms will begin by labeling the first vertex 0 and first edge 2, then use strictly increasing vertex labels that are less than the modulus, which induce strictly increasing edge labels that end with the modulus, thereby guaranteeing that the vertex labels and edge labels are distinct.

**Theorem 3.3.** The graph $2P_m \cup 2P_n \cup P_k$ is properly even harmonious.

**Proof.** The number of edges of the graph $2P_m \cup 2P_n \cup P_k$ is $q = 2m + 2n + k - 5$. The modulus is $2q = 4m + 4n + 2k - 10$. We may assume that $m \geq n$.

**Case 1:** $k$ is odd. See Figure 12.

**Step 1**
Label the first path $P_m$ starting with the first vertex with $0, 2, \ldots, 2m - 2$. The corresponding edge labels are $2, 6, \ldots, 4m - 6$.
Label the second $P_m$ starting with the first vertex with $1, 3, \ldots, 2m - 1$. The corresponding edge labels are $4, 8, \ldots, 4m - 4$.

**Step 2**
Label the first path $P_n$ starting with the first vertex with $2m + k - 3, 2m + k - 1, \ldots, 2m + 2n + k - 7, 2m + 2n + k - 5$. The corresponding edge labels are $4m + 2k - 4, 4m + 2k, \ldots, 4m + 4n + 2k - 12$.
Label the second $P_n$ starting with the first vertex with $2m + k - 2, 2m + k, \ldots, 2m + 2n + k - 6, 2m + 2n + k - 4$. The corresponding edge labels are $4m + 2k - 2, 4m + 2k + 2, \ldots, 4m + 4n + 2k - 10$.

**Step 3**
Label the path $P_k$ starting with the first vertex, and skipping a vertex each time, with $\frac{8m + 4n + k - 13}{2}, \frac{8m + 4n + k - 9}{2}, \ldots, \frac{8m + 4n + 3k - 15}{2}$. Wrap around
and continue with \(\frac{8m+4n+3k-11}{2}, \ldots, \frac{8m+4n+5k-17}{2}\). The corresponding edge labels are \(8m + 4n + 2k - 12, 8m + 4n + 2k - 10, \ldots, 8m + 4n + 4k - 16\).

\[0 \quad 2 \quad 6 \quad 10 \quad 14 \quad 18 \quad 22 \quad 26 \quad 14\]
\[1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 11 \quad 13 \quad 15 \quad 0\]
\[24 \quad 26 \quad 28 \quad 30 \quad 32 \quad 50 \quad 52 \quad 54 \quad 56 \quad 58 \quad 60 \quad 62 \quad 64 \quad 66 \quad 0\]
\[30 \quad 32 \quad 34 \quad 36 \quad 38 \quad 40 \quad 42 \quad 44 \quad 46 \quad 48 \quad 50 \quad 52 \quad 54 \quad 56 \quad 58 \quad 60 \quad 62 \quad 64 \quad 66 \quad 0\]

Figure 12: Properly even harmonious labeling of \(2P_3 \cup 2P_3 \cup P_{11} \pmod{64}\).

**Case 2:** \(k\) is even. See Figure 13.

**Step 1**
Label the first path \(P_m\) starting with the first vertex with 0, 2, \ldots, 2m - 2. The corresponding edge labels are 2, 6, \ldots, 4m - 6.
Label the second \(P_m\) starting with the first vertex with 1, 3, \ldots, 2m - 1. The corresponding edge labels are 4, 8, \ldots, 4m - 4.

**Step 2**
Label the first path \(P_n\) starting with the first vertex with 2m + k - 2, 2m + k, \ldots, 2m + 2n + k - 6, 2m + 2n + k - 4. The corresponding edge labels are 4m + 2k - 2, 4m + 2k + 2, \ldots, 4m + 4n + 2k - 10.
Label the second \(P_n\) starting with the first vertex with 2m + k - 3, 2m + k - 1, \ldots, 2m + 2n + k - 7, 2m + 2n + k - 5. The corresponding edge labels are 4m + 2k - 4, 4m + 2k, \ldots, 4m + 4n + 2k - 12.

**Step 3**
Label \(P_k\) starting with the first vertex, and skipping a vertex each time, with \(\frac{8m+4n+k-12}{2}, \frac{8m+4n+k-8}{2}, \ldots, \frac{8m+4n+3k-16}{2}\). Wrap around, starting with \(\frac{8m+4n+3k-12}{2}, \frac{8m+4n+3k-8}{2}, \ldots, \frac{8m+4n+5k-16}{2}\). The corresponding edge labels are \(8m + 4n + 2k - 12, 8m + 4n + 2k - 10, \ldots, 8m + 4n + 4k - 16\).
Theorem 3.4. The graph \(2P_m \cup 2P_n \cup C_k\), \((k \text{ odd})\) is properly even harmonious.

Proof. The number of edges of the graph \(2P_m \cup 2P_n \cup C_k\) is \(q = 2m + 2n + k - 4\). The modulus is \(2q = 4m + 4n + 2k - 8\). See Figures 14 and 15. We may assume that \(m \geq n\).

Step 1
Label the vertices of the first \(P_m\) starting with the first vertex with \(0, 2, \ldots, 2m - 2\). The corresponding edge labels are \(2, 6, \ldots, 4m - 6\).
Label the vertices of the second \(P_m\) starting with the first vertex with \(1, 3, \ldots, 2m - 1\). The corresponding edge labels are \(4, 8, \ldots, 4m - 4\).

Step 2
Label the vertices of the first \(P_n\) starting with the first vertex with \(2m + k - 1, 2m + k + 1, \ldots, 2m + 2n + k - 3\). The corresponding edge labels are \(4m + 2k, 4m + 2k + 4, \ldots, 4m + 4n + 2k - 8 = 0\).
Label the vertices of the second \(P_n\) starting with the first vertex with \(2m + k - 2, 2m + k, \ldots, 2m + 2n + k - 2\). The corresponding edge labels are \(4m + 2k, 4m + 2k + 2, \ldots, 4m + 4n + 2k - 10\).

Step 3
Label the vertices of \(C_n\) starting with the first vertex, skipping a vertex each time with \(\frac{8m + 4n + k - 9}{2}, \frac{8m + 4n + k - 5}{2}, \ldots, \frac{8m + 4n + 3k - 11}{2}\), wrap around and continue with \(\frac{8m + 4n + 3k - 7}{2}, \ldots, \frac{8m + 4n + 5k - 13}{2}\). The corresponding edge labels are \(4m, 4m + 2, \ldots, 4m + 2k - 4\). \(\square\)
Theorem 3.5. The graph $2P_m \cup 2P_n \cup C_4$ is properly even harmonious.

Proof. The number of edges of $2P_m \cup 2P_n \cup C_4$ is $q = 2m + 2n$. The modulus is $2q = 4m + 4n$. See Figures 16 and 17. We may assume that $m \geq n$.

**Step 1**
Label the first $P_m$ starting with the first vertex with $0, 2, \ldots, 2m - 2$. The corresponding edge labels are $2, 6, \ldots, 4m - 6$.
Label the second $P_m$ starting with the first vertex with $1, 3, \ldots, 2m - 1$. The corresponding edge labels are $4, 8, \ldots, 4m - 4$.

**Step 2**
Label the first $P_n$ starting with the first vertex with $2m + 2, 2m + 4, \ldots, 2m + 2n$. The corresponding edge labels are $4m + 6, 4m + 10, \ldots, 4m + 4n - 2$.
Label the second $P_n$ starting with the first vertex with $2m + 3, 2m + 5, \ldots, 2m + 2n + 1$. The corresponding edge labels are $4m + 8, 4m + 12, \ldots, 4m + 4n = 0$. 

Figure 14: Properly even harmonious labeling of $2P_9 \cup 2P_7 \cup C_7 \pmod{70}$.

Figure 15: Properly even harmonious labeling of $2P_8 \cup 2P_4 \cup C_9 \pmod{58}$. 

Step 3

Label the vertices of $C_4$ clockwise with $4m + 2n − 3, 4m + 2n + 1, 4m + 2n − 1, 4m + 2n + 5$. The corresponding edge labels are $8m + 4n − 2 = 4m − 2, 8m + 4n = 4m, 8m + 4n − 4 = 4m + 4$. 

\[ \begin{array}{ccccccccc}
2 & 6 & 10 & 14 & 18 & 22 & 26 & 30 & 34 & 38 & 42 & 46 & 50 & 54 & 58 & 62 \\
4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 & 44 & 48 & 52 & 56 & 60 & 64 \\
51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 & 87 & 91 & 95 & 99 & 103 & 107 & 111 \\
\end{array} \]

Figure 16: Properly even harmonious labeling of $2P_{12} \cup 2P_5 \cup C_4 \ (\text{mod} \ 60)$.

Our next result is an extension of the previous theorems.

**Theorem 3.6.** The graph $2P_m \cup 2P_n \cup 2K_4$ is properly even harmonious.

**Proof.** The number of edges of the graph $2P_m \cup 2P_n \cup 2K_4$ is $q = 2m + 2n + 8$. The modulus is $2q = 4m + 4n + 16$. We may assume that $m \geq n$. See Figure 18.

Step 1

Label the first $P_m$ starting with the first vertex with $0, 2, \ldots, 2m − 2$. The corresponding edge labels are $2, \ldots, 4m − 6$.

Label the second $P_m$ starting with the first vertex with $1, 3, \ldots, 2m − 1$. The corresponding edge labels are $4, 8, \ldots, 4m − 4$.
Step 2
Label the first \( P_n \) starting with the first vertex with \( 2m + 10, 2m + 12, \ldots, 2m + 2n + 8 \). The corresponding edge labels are \( 4m + 22, \ldots, 4m + 4n + 14 \).
Label the second \( P_n \) starting with the first vertex with \( 2m + 11, 2m + 13, \ldots, 2m + 2n + 9 \). The corresponding edge labels are \( 4m + 24, \ldots, 4m + 4n + 16 = 0 \).

Step 3
Label the vertices of the first \( K_4 \) clockwise with \( 4m + 2n + 6, 4m + 2n + 10, 4m + 2n + 8, 4m + 2n + 14 \). The corresponding edge labels are \( 8m + 4n + 16 = 4m, 8m + 4n + 18 = 4m + 2, 8m + 4n + 22 = 4m + 6, 8m + 4n + 20 = 4m + 4, 8m + 4n + 24 = 4m + 8 \).
Label the vertices of the other \( K_4 \) with \( 4m + 2n + 11, 4m + 2n + 15, 4m + 2n + 19, 4m + 2n + 17 \). The corresponding edge labels are \( 8m + 4n + 26 = 4m + 10, 8m + 4n + 34 = 4m + 18, 8m + 4n + 36 = 4m + 20, 8m + 4n + 28 = 4m + 12, 8m + 4n + 30 = 4m + 14, 8m + 4n + 32 = 4m + 16 \).

\[ \square \]

Figure 18: Properly even harmonious labeling of \( 2P_9 \cup 2P_5 \cup 2K_4 \) (mod 72).

Definition 3.7. Let \( G_1 \) and \( G_2 \) be two graphs with no vertex in common. We define the join of \( G_1 \) and \( G_2 \), denoted by \( G_1 + G_2 \), to be the graph which consists of edges which join every vertex of \( G_1 \) to every vertex of \( G_2 \).

Definition 3.8. The wheel, \( W_n \) \((n \geq 3)\), is the join \( K_1 + C_n \).

Theorem 3.9. The graph \( 2P_m \cup 2P_n \cup 2W_4 \) is properly even harmonious.

Proof. The number of edges of the graph \( 2P_m \cup 2P_n \cup W_4 \) is \( q = 2m + 2n + 12 \). The modulus is \( 2q = 4m + 4n + 24 \). We may assume that \( m \geq n \). See Figure 19.

Step 1
Label the first \( P_m \) starting with the first vertex with 0, 2, \ldots, 2m − 2. The corresponding edge labels are 2, \ldots, 4m − 6.
Label the second $P_m$ starting with the first vertex with $1, 3, \ldots, 2m - 1$. The corresponding edge labels are $4, 8, \ldots, 4m - 4$.

**Step 2**
Label the first $P_n$ starting with the first vertex with $2m + 14, 2m + 12, \ldots, 2m + 2n + 12$. The corresponding edge labels are $4m + 30, \ldots, 4m + 4n + 22$.
Label the second $P_n$ starting with the first vertex with $2m + 15, 2m + 17, \ldots, 2m + 2n + 13$. The corresponding edge labels are $4m + 2, \ldots, 4m + 4n + 24 = 0$.

**Step 3**
Label the vertices of the first $W_4$ clockwise with $4m + 2n + 10, 4m + 2n + 14, 4m + 2n + 20, 4m + 2n + 12$, and label the center of the wheel with $4m + 2n + 16$. The corresponding edge labels are $8m + 4n + 24 = 4m, 8m + 4n + 34 = 4m + 10, 8m + 4n + 42 = 4m + 8, 8m + 4n + 22 = 4m - 2$.
The labels on the diagonals are $8m + 4n + 26 = 4m + 2, 8m + 4n + 30 = 4m + 6, 8m + 4n + 36 = 4m + 12, 8m + 4n + 28 = 4m + 4$.
Label the vertices of the other $W_4$ with $4m + 2n + 17, 4m + 2n + 23, 4m + 2n + 27, 4m + 2n + 25$, and label the center of the wheel with $4m + 2n + 21$. The corresponding edge labels are $8m + 4n + 40 = 4m + 16, 8m + 4n + 50 = 4m + 26, 8m + 4n + 52 = 4m + 28, 8m + 4n + 42 = 4m + 18$, and the labels on the diagonals are $8m + 4n + 38 = 4m + 14, 8m + 4n + 44 = 4m + 20, 8m + 4n + 48 = 4m + 24, 8m + 4n + 46 = 4m + 22$.

\[ \begin{array}{cccccccccccc}
0 & 2 & 6 & 4 & 10 & 14 & 18 & 22 & 12 \\
2 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 & 44 & 48 \\
5 & 9 & 11 & 13 & 58 & 62 & 66 & 30 & 34 & 38 & 42 & 46 \\
29 & 31 & 33 & 35 & 53 & 57 & 61 & 63 & 50 & 56 \\
\end{array} \]

Figure 19: Properly even harmonious labeling of $2P_7 \cup 2P_4 \cup 2W_4$ (mod 68).

**Theorem 3.10.** The graph $2P_m \cup 2P_n \cup 2C_k$ ($k$ is odd) is properly even harmonious.
Proof. The number of edges of the graph $2P_m \cup 2P_n \cup 2C_k$ is $q = 2m + 2n + 2k - 4$. The modulus is $2q = 4m + 4n + 4k - 8$. We may assume that $m \geq n$. See Figures 20 and 21.

**Step 1**
Label the first path $P_m$ starting with the first vertex with $0, 2, \ldots, 2m - 2$. The corresponding edge labels are $2, 6, \ldots, 4m - 6$.
Label the second $P_m$ starting with the first vertex with $1, 3, \ldots, 2m - 1$. The corresponding edge labels are $4, 8, \ldots, 4m - 4$.

**Step 2**
Label the first path $P_n$ starting with the first vertex with $2m + 2k - 2, 2m + 2k, \ldots, 2m + 2k + 2n - 6, 2m + 2k + 2n - 4$. The corresponding edge labels are $4m + 4k - 2, 4m + 4k + 2, \ldots, 4m + 4k + 4n - 10$.
Label the second $P_n$ starting with the first vertex with $2m + 2k - 1, 2m + 2k + 1, \ldots, 2m + 2k + 2n - 5, 2m + 2k + 2n - 3$. The corresponding edge labels are $4m + 4k, 4m + 4k + 4, \ldots, 4m + 4k + 4n - 8$.

**Step 3**
Label the first $C_k$ starting with the first vertex, and skipping a vertex each time, with $\frac{8m + 4n + 3k - 9}{2}, \frac{8m + 4n + 3k - 5}{2}, \ldots, \frac{8m + 4n + 4k - 11}{2}$. Wrap around and continue with $\frac{8m + 4n + 4k - 7}{2}, \frac{8m + 4n + 4k - 3}{2}, \ldots, \frac{8m + 4n + 7k - 13}{2}$. The corresponding edge labels are $8m + 4n + 4k - 8 = 4m, 8m + 4n + 4k - 6 = 4m + 2, \ldots, 8m + 4n + 6k - 12 = 4m + 2k - 4$, and the label on the arc is $4m - 2$.
Label the second $C_k$ starting with the first vertex with $\frac{8m + 4n + 5k - 9}{2}, \frac{8m + 4n + 5k - 5}{2}, \ldots, \frac{8m + 4n + 7k - 11}{2}$. Wrap around and continue with $\frac{8m + 4n + 4k + 7k - 7}{2}, \frac{8m + 4n + 4k + 7k - 3}{2}, \ldots, \frac{8m + 4n + 9k - 13}{2}$. The corresponding edge labels are $8m + 4n + 6k - 8 = 4m + 2k, 8m + 4n + 6k - 6 = 4m + 2k + 2, \ldots, 8m + 4n + 8k - 12 = 4m + 4k - 4$.

![Figure 20: Properly even harmonious labeling of $2P_{10} \cup 2P_7 \cup 2C_7$ (mod 88).](image-url)
Gallian and Schoenhard proved that $F_n \cup C_4$ ($F_n$ is the fan $P_n + K_1$) is properly even harmonious for $n > 1$. Our next two results handle $F_n \cup K_4$ and $F_n \cup 2K_4$.

**Theorem 3.11.** The graph $F_n \cup K_4$ is properly even harmonious, where $n \geq 2$.

*Proof.* The number of edges of $F_n \cup K_4$ is $q = 2n + 5$. The modulus is $2q = 4n + 10$.

**Case 1:** $n = 0 \pmod{4}$. See Figure 22.

**Step 1**
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n - 1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 1, n + 3, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n - 1$. The edge labels corresponding to the path are $n + 2, n + 4, \ldots, 3n - 2$. The remaining edge labels are $3n, 4n, 3n + 2, 4n + 2, 3n + 4, \ldots, 4n - 2, 5n - 2$ and label the rest of its vertices as $0, 2, \ldots, 2m - 4$.

**Step 2**
Label the vertices of $K_4$ clockwise as $5n - 4, 5n + 4, 5n + 12, 5n + 8$. The corresponding edge labels are $5n = n - 10, 5n + 8 = n - 2, n, 5n + 2 = n - 8$, the main diagonal label is $5n + 6 = n - 4$, and the other diagonal label is $5n + 4 = n - 6$. 

Figure 21: Properly even harmonious labeling of $2P_8 \cup 2P_4 \cup 2C_5 \pmod{60}$. 


Figure 22: Properly even harmonious labeling of $F_{12} \cup K_4 \ (\text{mod} \ 58)$.

Case 2: $n = 1 \ (\text{mod} \ 4)$. See Figure 23.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2$, $n + 4$, \ldots, $2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3$, $n + 5$, \ldots, $3n - 1$. The remaining edge labels are $3n + 1$, $4n + 2$, $3n + 3$, $4n + 2$, $3n + 4$, \ldots, $5n - 1$, $4n$.

Step 2
Label the vertices of $K_4$ clockwise as $\frac{5n - 1}{2}$, $\frac{5n + 15}{2}$, $\frac{5n + 3}{2}$, $\frac{5n + 7}{2}$. The corresponding edge labels are $5n + 7 = n - 3$, $5n + 9 = n - 1$, $5n + 5 = n - 5$, $5n + 3 = n - 7$. Label the main diagonal with $5n + 11 = n + 1$ and the other diagonal with $5n + 1 = n - 9$.

Figure 23: Properly even harmonious labeling of $F_{13} \cup K_4 \ (\text{mod} \ 62)$.

Case 3: $n = 2 \ (\text{mod} \ 4)$. See Figure 24.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n - 1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 1$, $n + 3$, \ldots, $2n - 1$. Label the vertex of degree $n$ with $3n - 1$. The edge labels corresponding to the path are $n + 2$, $n + 4$, \ldots, $3n - 2$. The remaining edge labels are $3n$, $4n$, $3n + 2$, $4n + 2$, $3n + 4$, \ldots, $4n - 2$, $5n - 2$. 
Step 2
Label the vertices of $K_4$ clockwise with $\frac{5n-2}{2}, \frac{5n+14}{2}, \frac{5n+2}{2}, \frac{5n+6}{2}$. The corresponding edge labels are $5n + 6 = n - 4, 5n + 8 = n - 2, 5n + 4 = n - 6, 5n + 2 = n - 8$. Label the main diagonal with $n$, and the other diagonal with $5n = n - 10$.

Figure 24: Properly even harmonious labeling of $F_{10} \cup W_4$ (mod 54).

Case 4: $n = 3 \pmod{4}$.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 14n$.

Step 2
Label the vertices of $K_4$ clockwise as $\frac{5n-3}{2}, \frac{5n+5}{2}, \frac{5n+13}{2}, \frac{5n+9}{2}$. The corresponding edge labels are $5n + 1, 5n + 9 = n - 1, 5n + 11 = n + 1, 5n + 2 = n - 8, 5n + 3$. Label the main diagonal with $5n + 7 = n - 3$, and the other diagonal with $5n + 5 = n - 5$.

Since the vertex labels are strictly increasing and less than the modulus, they are distinct. The edge labels in $F_n$ start at $n + 2$ or $n + 3$ and increase by increments of 2 so they are distinct. Finally, the edge labels in $K_4$ begin with an edge that is 2 more than the largest edge label in $F_n$ and increase by 2s until ending $n$ or $n + 1$. So, there is no overlapping of labels.

Theorem 3.12. The graph $F_n \cup W_4$ is properly even harmonious.

Proof. The number of edges of the graph $F_n \cup W_4$ is $q = 2n + 7$. The modulus is $2q = 4n + 14$. 

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Case 1: $n = 0 \pmod{4}$. See Figure 25.

Step 1
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n - 1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 1, n + 3, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n - 1$. The edge labels corresponding to the path are $n + 2, n + 4, \ldots, 3n - 2$. The remaining edge labels are $3n, 4n, 3n + 2, 4n + 2, 3n + 4, \ldots, 4n - 2, 5n - 2$.

Step 2
Label the vertices of $W_4$ clockwise with $\frac{5n - 4}{2}, \frac{5n + 8}{2}, \frac{5n + 16}{2}, \frac{5n + 12}{2}$, and the center of the wheel with $\frac{5n + 4}{2}$, the labels are assigned clockwise. The corresponding edge labels are $5n + 2 = n - 12, 5n + 12 = n - 2, 5n + 14 = n, 5n + 4 = n - 10, 5n = n - 14, 5n + 6 = n - 8, 5n + 10 = n - 4, 5n + 8 = n - 6$.

Figure 25: Properly even harmonious labeling of $F_8 \cup W_4 \pmod{46}$.

Case 2: $n = 1 \pmod{4}$. See Figure 26.

Step 1
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 2, 3n + 4, \ldots, 5n - 1, 4n$.

Step 2
Label the vertices of $W_4$ clockwise with $\frac{5n - 1}{2}, \frac{5n + 7}{2}, \frac{5n + 19}{2}, \frac{5n + 3}{2}$. The corresponding edge labels on the rim are $5n + 3 = n - 11, 5n + 13, 5n + 11, 5n + 1 = n - 13$. On the diagonals, the labels are $5n + 5 = n - 9, 5n + 9 = n - 5, 5n + 15 = n + 1, 5n + 7 = n - 7$. 

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Case 3: $n = 2 \pmod{4}$. See Figure 27.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n-1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n+1, n+3, \ldots, 2n-1$. Label the vertex of degree $n$ with $3n-1$. The edge labels corresponding to the path are $n+2, n+4, \ldots, 3n-2$. The remaining edge labels are $3n, 4n, 3n+2, 4n+2, 3n+4, \ldots, 4n-2, 5n-2$.

Step 2
Label the vertices of $W_4$ clockwise as $\frac{5n-2}{2}, \frac{5n+6}{2}, \frac{5n+18}{2}, \frac{5n+2}{2}$. The corresponding edge labels are $5n+2 = n-12, 5n+12 = n-2, 5n+10 = n-4, 5n = n-14$. The labels on the diagonal are $5n+4 = n-10, 5n+8 = n-6, 5n+14 = n, 5n+6 = n-8$.

Case 4: $n = 3 \pmod{4}$. See Figure 28.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n+2, n+4, \ldots, 2n-1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n+3, n+5, \ldots, 3n-1$. The remaining edge labels are $3n+1, 4n+2, 3n+3, 4n+4, 3n+5, \ldots, 5n-1, 4n$. 

Figure 26: Properly even harmonious labeling of $F_3 \cup W_4 \pmod{50}$.

Figure 27: Properly even harmonious labeling of $F_{10} \cup W_4 \pmod{54}$.
Step 2
Label the vertices of $W_4$ clockwise with $\frac{5n-3}{2}, \frac{5n+9}{2}, \frac{5n+17}{2}, \frac{5n+13}{2}$. The center of the wheel is labeled with $\frac{5n+5}{2}$. The corresponding edge labels are $\frac{5n+3}{2}, \frac{5n+13}{2}, \frac{5n+15}{2}, \frac{5n+5}{2}, \frac{5n+1}{2}, \frac{5n+7}{2}, \frac{5n+11}{2}, 5n + 9$.

![Figure 28: Properly even harmonious labeling of $F_{11} \cup W_4$ (mod 58).](image)

An argument analogous to that given in the proof of Theorem 3.9 shows that the vertex labels and edge labels are distinct.

**Theorem 3.13.** The graph $F_n \cup 2K_4$ is properly even harmonious.

**Proof.** The number of edges of $F_n \cup 2K_4$ is $q = 2n + 11$. The modulus is $2q = 4n + 22$.

**Case 1:** $n = 0 \pmod{4}$. See Figure 29.

Step 1
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n - 1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 1, n + 3, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n - 1$. The edge labels corresponding to the path are $n + 2, n + 4, \ldots, 3n - 2$. The remaining edge labels are $3n, 4n, 3n + 2, 4n + 2, 3n + 4, \ldots, 4n - 2, 5n - 2$.

Step 2
Label the vertices of one of the $K_4$ clockwise with $\frac{5n-4}{2}, \frac{5n+4}{2}, \frac{5n+12}{2}, \frac{5n+8}{2}$. The corresponding edge labels are $5n, 5n + 8, 5n + 10, 5n + 2$, the label on the main diagonal is $5n + 6$, and on the other diagonal is $5n + 4$.

Step 3
Label the vertices of the other $K_4$ clockwise with $\frac{9n+32}{2}, \frac{9n+48}{2}, \frac{9n+36}{2}, \frac{9n+40}{2}$. 

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The corresponding edge labels are $5n + 18 = n - 4, 5n + 20 = n - 2, 5n + 16 = n - 6, 5n + 14 = n - 8$, the label on the main diagonal is $5n + 12 = n - 10$, and the other diagonal is $5n + 22 = n$.

Figure 29: Properly even harmonious labeling of $F_8 \cup 2K_4 \, (\text{mod } 54)$.

**Case 2:** $n = 1 \, (\text{mod } 4)$. See Figure 30.

**Step 1**
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 2, 3n + 4, \ldots, 5n - 1, 4n$.

**Step 2**
Label the vertices of $K_4$ clockwise with $\frac{5n - 1}{2}, \frac{5n + 15}{2}, \frac{5n + 3}{2}, \frac{5n + 7}{2}$. The corresponding edge labels are $5n + 7 = n - 15, 5n + 9 = n - 13, 5n + 5 = n - 17, 5n + 3 = n - 19$. The label on the main diagonal is $5n + 1 = n - 21$ and on the other diagonal is $5n + 11 = n - 11$.

**Step 3**
Label the vertices of the other $K_4$ clockwise as $\frac{2q + 5n + 9}{2}, \frac{2q + 5n + 17}{2}, \frac{2q + 5n + 25}{2}, \frac{2q + 5n + 21}{2}$. The corresponding edge labels are $5n + 13 = n - 9, 5n + 21 = n - 15, 5n + 23 = n + 1, 5n + 15 = n - 7$, the label on the main diagonal is $5n + 17$, and on the other diagonal it is $5n + 19 = n - 3$. 

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Case 3: $n = 2 \pmod{4}$.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n-1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n+1, n+3, \ldots, 2n-1$. Label the vertex of degree $n$ with $3n-1$. The edge labels corresponding to the path are $n+2, n+4, \ldots, 3n-2$. The remaining edge labels are $3n, 4n, 3n+2, 4n+2, 3n+4, \ldots, 4n-2, 5n-2$.

Step 2
Label the vertices of $K_4$ clockwise with $\frac{5n-2}{2}, \frac{5n+14}{2}, \frac{5n+2}{2}, \frac{5n+6}{2}$. The corresponding edge labels are $5n+6 = n-16, 5n+8 = n-14, 5n+4 = n-18, 5n+2 = n-20$. The main is diagonal $5n$, and the other diagonal is $5n+10 = n-12$.

Step 3
Label the vertices of the other $K_4$ clockwise with $\frac{2q+5n-8}{2}, \frac{2q+5n+16}{2}, \frac{2q+5n+24}{2}$, and $\frac{2q+5n+20}{2}$. The corresponding edge labels are $5n+12 = n-10, 5n+20 = n-2, 5n = n-22, 5n+22 = n$. The label on the main diagonal is $5n+16 = n-6$, and on the second diagonal is $5n+18 = n-4$.

Case 4: $n = 3 \pmod{4}$. See Figure 31.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n+2, n+4, \ldots, 2n-1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n+3, n+5, \ldots, 3n-1$. The remaining edge labels are $3n+1, 4n+2, 3n+3, 4n+4, 3n+5, \ldots, 5n-1, 4n$. 

Figure 30: Properly even harmonious labeling of $F_9 \cup 2K_4 \pmod{58}$. 

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Step 2
Label the vertices of one of the $K_4$ with $\frac{5n-3}{2}, \frac{5n+5}{2}, \frac{5n+13}{2}, \frac{5n+9}{2}$, the labels are assigned clockwise. The corresponding edge labels are $5n+1, 5n+9, 5n+11, 5n+3, 5n+5$. The main diagonal is $5n+5$, and the other diagonal is $5n+7$.

Step 3
Label the vertices of the other $K_4$ with $\frac{9n+33}{2}, \frac{9n+49}{2}, \frac{9n+37}{2}, \frac{9n+41}{2}$, the labels are assigned clockwise. The corresponding edge labels are $5n+19, 5n+20 = n-2, 5n+21, 5n+17, 5n+15$, the main diagonal is $5n+13 = n-9$, and the other diagonal is $5n+23 = n+1$.

\[ \square \]

Figure 31: Properly even harmonious labeling of $F_7 \cup 2K_4 \pmod{50}$.

Since the vertex labels and edge labels in $F_n$ and the first $K_4$ have opposite parity and are strictly increasing and less than the modulus, they are distinct. The vertex labels of the second $K_4$ are distinct by construction and if a vertex label $x$ of the second $K_4$ overlapped with a vertex label $y$ in the first $K_4$ then $x + 2q = y$. But simple algebra shows that this does not hold.

**Theorem 3.14.** The graph $F_n \cup 2W_4$ is properly even harmonious.

**Proof.** The number of edges of the graph $F_n \cup 2W_4$ is $q = 2n + 15$. The modulus is $2q = 4n + 30$.

**Case 1:** $n = 0 \pmod{4}$. See Figure 32.

**Step 1**
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n-1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n+1, n+3, \ldots, 2n-1$. Label the vertex of degree $n$ with $3n-1$. The edge labels corresponding to the path are $n+2, n+4, \ldots, 3n-2$.
The remaining edge labels are $3n, 4n, 3n+2, 4n+2, 3n+4, \ldots, 4n-2, 5n-2$.

**Step 2**
Label the vertices of one of the $W_4$ clockwise as $\frac{5n-4}{2}, \frac{5n+8}{2}, \frac{5n+16}{2}, \frac{5n+12}{2}$ and the center of the wheel as $\frac{5n+4}{2}$. The corresponding edge labels are $5n+2 = n-28, 5n+12 = n-18, 5n+14 = n-16, 5n+4 = n-26, 5n = n-30, 5n+6 = n-24, 5n+10 = n-20, 5n+8 = n-22$.

**Step 3**
Label the vertices of the other $W_4$ clockwise as $\frac{5n+14}{2}, \frac{5n+22}{2}, \frac{5n+34}{2}, \frac{5n+18}{2}$, and the center of the wheel is $\frac{5n+26}{2}$. The corresponding edge labels on the rim are $5n+18 = n-12, 5n+28 = n-2, 5n+26 = n-4, 5n+16 = n-14$ and the labels on the diagonals are $5n+20 = n-10, 5n+24 = n-6, 5n+30 = n, 5n+22 = n-8$.

![Figure 32: Properly even harmonious labeling of $F_8 \cup 2W_4$ (mod 62).](image)

**Case 2:** $n = 1 \pmod{4}$. See Figure 33.

**Step 1**
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n+2, n+4, \ldots, 2n-1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n+3, n+5, \ldots, 3n-1$. The remaining edge labels are $3n+1, 4n+2, 3n+3, 4n+2, 3n+4, \ldots, 5n-1, 4n$.

**Step 2**
Label the vertices of one of the $W_4$ clockwise as $\frac{5n-1}{2}, \frac{5n+7}{2}, \frac{5n+19}{2}, \frac{5n+3}{2}$. Label the center of the wheel with $\frac{5n+11}{2}$. The corresponding edge labels are $5n+3, 5n+23, 5n+21, 5n+1$. On the diagonals, the labels are $5n+5, 5n+9, 5n+15, 5n+7$. 24
Step 3
Label the vertices of the other $W_4$ clockwise with $\frac{5n+13}{2}$, $\frac{5n+25}{2}$, $\frac{5n+33}{2}$, $\frac{5n+29}{2}$. The center of the wheel is labeled with $\frac{5n+21}{2}$. The corresponding edge labels are $5n+19$, $5n+29$, $5n+31$, $5n+21$. On the diagonals, the labels are $5n+17$, $5n+23$, $5n+27$, $5n+25$.

![Diagram](image)

Figure 33: Properly even harmonious labeling of $F_9 \cup 2W_4 \pmod{62}$.

Case 3: $n = 2 \pmod{4}$. See Figure 33.

Step 1
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n - 1$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 1, n + 3, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n - 1$. The edge labels corresponding to the path are $n + 2, n + 4, \ldots, 3n - 2$. The remaining edge labels are $3n, 4n, 3n + 2, 4n + 2, 3n + 4, \ldots, 4n - 2, 5n - 2$.

Step 2
Label the vertices of $W_4$ clockwise with $\frac{5n-2}{2}$, $\frac{5n+6}{2}$, $\frac{5n+18}{2}$, $\frac{5n+2}{2}$. Label the center of the wheel as $\frac{5n+10}{2}$. The corresponding edge labels are $5n + 2 = n - 28, 5n + 12 = n - 18, 5n + 10 = n - 20, 5n = n - 30$. The labels on the diagonal are $5n + 4 = n - 26, 5n + 8 = n - 22, 5n + 14 = n - 16, 5n + 6 = n - 24$.

Step 3
Label the vertices of the other $W_4$ clockwise with $\frac{5n+12}{2}, \frac{5n+24}{2}, \frac{5n+32}{2}, \frac{5n+28}{2}$. The center of the wheel is $\frac{5n+20}{2}$. The corresponding edge labels are $5n + 18 = n - 12, 5n + 28 = n - 2, 5n + 30 = n, 5n + 20 = n - 10, 5n + 16 = n - 14, 5n + 22 = n - 8, 5n + 26 = n - 4, 5n + 24 = n - 6$. 
Figure 34: Properly even harmonious labeling of $F_{10} \cup 2W_4 \pmod{70}$.

Case 4: $n = 3 \pmod{4}$. See Figure 35.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, . . . , $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n$.

Step 2
Label the vertices of one of the $W_4$ clockwise with $\frac{5n + 9}{2}, \frac{5n + 17}{2}, \frac{5n + 13}{2}$. The center of the wheel is labeled with $\frac{5n + 5}{2}$. The corresponding edge labels are $5n + 3 = n - 27, 5n + 13 = n - 17, 5n + 15 = n - 15, 5n + 5 = n - 25, 5n + 1 = n - 29, 5n + 7 = n - 23, 5n + 11 = n - 19, 5n + 9 = n - 21$.

Step 3
Label the vertices of the other $W_4$ clockwise with $\frac{5n + 15}{2}, \frac{5n + 23}{2}, \frac{5n + 35}{2}, \frac{5n + 19}{2}$, and the center of the wheel with $\frac{5n + 27}{2}$. The corresponding edge labels are $5n + 19 = n - 11, 5n + 29 = n - 1, 5n + 27 = n - 3, 5n + 21 = n - 9, 5n + 25 = n - 5, 5n + 31 = n + 1, 5n + 23 = n - 7$.

Figure 35: Properly even harmonious labeling of $F_7 \cup 2W_4 \pmod{58}$.

\[\square\]

Theorem 3.15. For $n \geq 3$ odd, the graph $W_n \cup K_4$ is properly even.
Proof. The number of edges of the graph $W_n \cup K_4$ is $q = 2n + 6$. The modulus is $2q = 4n + 12$.

Case 1: $n = 1 \pmod{4}$. See Figure 36.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, . . . , $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1, n + 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n$.

Step 2
Label the vertices of $K_4$ clockwise with $\frac{5n - 1}{2}, \frac{5n + 15}{2}, \frac{5n + 3}{2}, \frac{5n + 7}{2}$. The corresponding edge labels are $5n + 7 = n - 5, 5n + 9 = n - 3, 5n + 5 = n - 7, 5n + 3 = n - 9$. The label on the main diagonal is $5n + 1 = n - 11$ and on the other diagonal is $5n + 11 = n - 1$.

Figure 36: Properly even harmonious labeling of $W_9 \cup K_4 \pmod{48}$.

Case 2: $n = 3 \pmod{4}$. See Figure 37.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, . . . , $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1, n + 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n$.

Step 2
Label the vertices of $K_4$ clockwise as $\frac{5n - 3}{2}, \frac{5n + 5}{2}, \frac{5n + 13}{2}, \frac{5n + 9}{2}$. The corre-
sponding edge labels are $5n + 1 = n - 11, 5n + 9 = n - 3, 5n + 11 = n - 1, 5n + 3 = n - 9$, the label on the main diagonal is $5n + 5$, and on the other diagonal is $5n + 7 = n - 5$.

\[ \begin{align*}
5n + 3 &= n - 13, \\
5n + 11 &= n - 5, \\
5n + 1 &= n - 15.
\end{align*} \]

Figure 37: Properly even harmonious labeling of $W_7 \cup K_4 \pmod{40}$.

**Theorem 3.16.** For $n \geq 3$ odd, $W_n \cup W_4$ is properly even harmonious.

**Proof.** The number of edges of the graph $W_n \cup W_4$ is $q = 2n + 8$. The modulus is $2q = 4n + 16$.

**Case 1:** $n = 1 \pmod{4}$. See Figure 38.

**Step 1**
Label $W_n$ as shown in the case 1 of $W_n \cup K_4$.

**Step 2**
Label $W_4$ starting from the far top left corner and moving clockwise with $\frac{5n - 1}{2}, \frac{5n + 7}{2}, \frac{5n + 19}{2}, \frac{5n + 3}{2}$. Label the center of the wheel with $\frac{5n + 11}{2}$. The corresponding edge labels of the rim are $5n + 3 = n - 13, 5n + 13 = n - 3, 5n + 11 = n - 5, 5n + 1 = n - 15$. On the diagonals we have $5n + 5 = n - 11, 5n + 9 = n - 7, 5n + 15 = n - 1, 5n + 7 = n - 9$.

\[ \begin{align*}
5n + 1 &= n - 15, \\
5n + 9 &= n - 11, \\
5n + 11 &= n - 7, \\
5n + 13 &= n - 3.
\end{align*} \]

Figure 38: Properly even harmonious labeling of $W_9 \cup W_4 \pmod{52}$. 

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Case 2: \( n = 3 \pmod{4} \). See Figure 39.

**Step 1**
Label \( W_n \) as shown in the case 2 of \( W_n \cup K_4 \).

**Step 2**
Label \( W_4 \) starting from the far top left corner and moving clockwise with \( \frac{5n-3}{2}, \frac{5n+9}{2}, \frac{5n+17}{2}, \frac{5n+13}{2} \). Label the center of the wheel with \( \frac{5n+5}{2} \). The corresponding edge labels of the rim are \( 5n + 3 = n - 13, 5n + 13 = n - 3, 5n + 15 = n - 1, 5n + 5 = n - 11 \). On the diagonals we have \( 5n + 1 = n - 15, 5n + 7 = n - 9, 5n + 11 = n - 5, 5n + 9 = n - 7 \).

Figure 39: Properly even harmonious labeling of \( W_{11} \cup W_4 \pmod{60} \).

**Theorem 3.17.** For \( n \geq 3 \) odd, \( W_n \cup 2K_4 \) is properly even harmonious.

**Proof.** The number of edges of the graph \( W_n \cup 2K_4 \) is \( q = 2n + 12 \). The modulus is \( 2q = 4n + 24 \).

**Case 1:** \( n = 1 \pmod{4} \). See Figure 40.

**Step 1**
Label \( P_n \) starting with the first vertex with 1, 3, \ldots, \( n \) skipping a vertex each time. Wrap around and label the remaining vertices with \( n + 2, n + 4, \ldots, 2n - 1 \). Label the vertex of degree \( n \) with \( 3n \). The edge labels corresponding to the path are \( 3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n \).

**Step 2**
Label the vertices of one of the \( K_4 \) clockwise as \( \frac{5n-1}{2}, \frac{5n+15}{2}, \frac{5n+3}{2}, \frac{5n+7}{2} \). The corresponding edge labels are \( 5n + 7 = n - 17, 5n + 9 = n - 15, 5n + 5 = n - 19, 5n + 3 = n - 21 \). The label on the first diagonal is \( 5n + 1 = n - 23 \).
and on the second diagonal it is \(5n + 11 = n - 13\).

**Step 3**

Label the vertices of the other \(K_4\) clockwise as \(\frac{2q+5n+7}{2}, \frac{2q+5n+19}{2}, \frac{2q+5n+27}{2}\). The corresponding edge labels are \(5n + 13 = n - 11, 5n + 21 = n - 3, 5n + 23 = n - 1, 5n + 15 = n - 9, 5n + 17 = n - 7, 5n + 19 = n - 5\).

![Figure 40: Properly even harmonious labeling of \(W_9 \cup 2K_4\) (mod 60).](image)

**Case 2: n = 3 (mod 4).** See Figure 41.

**Step 1**

Label \(P_n\) starting with the first vertex with 1, 3, \ldots, \(n\) skipping a vertex each time. Wrap around and label the remaining vertices with \(n + 2, n + 4, \ldots, 2n - 1\). Label the vertex of degree \(n\) with \(3n\). The edge labels corresponding to the path are \(n + 3, n + 5, \ldots, 3n - 1, n + 1\). The remaining edge labels are \(3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n\).

**Step 2**

Label the vertices of one of the \(K_4\) clockwise as \(\frac{5n-3}{2}, \frac{5n+5}{2}, \frac{5n+11}{2}, \frac{5n+9}{2}\). The corresponding edge labels are \(5n + 1 = n - 23, 5n + 9 = n - 15, 5n + 11 = n - 13, 5n + 3 = n - 21\). The label on the main diagonal is \(5n + 5 = n - 19\) and on the other diagonal is \(5n + 7 = n - 17\).

**Step 3**

Label the vertices of the other \(K_4\) clockwise as \(\frac{5n+11}{2}, \frac{5n+27}{2}, \frac{5n+15}{2}, \frac{5n+19}{2}\). The corresponding labels are \(5n + 19 = n - 5, 5n + 21 = n - 3, 5n + 17 = n - 7, 5n + 15 = n - 9, 5n + 13 = n - 11, 5n + 23 = n - 1\).

**Theorem 3.18.** For \(n \geq 3\) odd, \(W_n \cup 2W_4\) is properly even harmonious.

**Proof.** The number of edges of the graph \(W_n \cup 2W_4\) is \(q = 2n + 16\). The modulus is \(2q = 4n + 32\). □
Case 1: \( n = 1 \pmod{4} \). See Figure 42.

Step 1
Label the \( W_n \) as shown in the case 1 of \( W_n \cup W_4 \).

Step 2
Label one of the \( W_4 \) starting from the far top left corner and moving clockwise with \( \frac{5n-1}{2} \), \( \frac{5n+19}{2} \), \( \frac{5n+3}{2} \). Label the center of the wheel with \( \frac{5n+11}{2} \). The corresponding edge labels on the rim are \( 5n + 3 \), \( 5n + 13 \), \( 5n + 11 \), \( 5n + 1 \), and the labels on the diagonals are \( 5n + 5 \), \( 5n + 9 \), \( 5n + 15 \), \( 5n + 10 \).

Step 3
Label the other \( W_4 \) starting from the far top left corner and moving clockwise with \( \frac{9n+47}{2} \), \( \frac{9n+55}{2} \), \( \frac{9n+67}{2} \), \( \frac{9n+51}{2} \). Label the center of the wheel with \( \frac{9n+59}{2} \). The corresponding edge labels on the rim are \( 9n + 51 = 5n + 19 \), \( 9n + 61 = 5n + 29 \), \( 9n + 59 = 5n + 27 \), \( 9n + 49 = 5n + 17 \). The labels on the diagonals are \( 9n + 53 = 5n + 21 \), \( 9n + 57 = 5n + 25 \), \( 9n + 63 = 5n + 31 \), \( 9n + 55 = 5n + 23 \).

Case 2: \( n = 3 \pmod{4} \). See Figure 42.

Step 1
Label the \( W_n \) as shown in the case 2 of \( W_n \cup K_4 \).
Step 2
Label $W_4$ starting from the far top left corner and moving clockwise with $\frac{5n-3}{2}, \frac{5n+9}{2}, \frac{5n+17}{2}, \frac{5n+13}{2}$. Label the center of the wheel with $\frac{5n+5}{2}$. The corresponding edge labels are $5n + 3, 5n + 13, 5n + 15, 5n + 5$, on the diagonals we have $5n + 1, 5n + 7, 5n + 11, 5n + 9$.

Step 3:
Label the other $W_4$ starting from the far top left corner and moving clockwise with $\frac{9n+45}{2}, \frac{9n+57}{2}, \frac{9n+65}{2}, \frac{9n+61}{2}$. Label the center of the wheel with $\frac{9n+53}{2}$. The corresponding edge labels are $9n + 51 = 5n + 19, 9n + 61 = 5n + 29, 9n + 63 = 5n + 31, 9n + 53 = 5n + 21$, and the labels on the diagonals are $9n + 49 = 5n + 17, 9n + 55 = 5n + 23, 9n + 59 = 5n + 27, 9n + 57 = 5n + 25$.

Figure 43: Properly even harmonious labeling of $W_7 \cup 2W_4$ (mod 60).

Definition 3.19. The complement $\overline{G}$ of a graph $G$ with $n$ vertices is the graph obtained from the complete graph $K_n$ by deleting all the edges of $G$.

Definition 3.20. The double cone $DC_n$, is $C_n + K_2$ ($n \geq 3$).

Theorem 3.21. For $n \geq 3$ odd, the graph $DC_n \cup K_4$ is properly even harmonious.

Proof. The number of edges of the graph $DC_n \cup K_4$ is $q = 3n + 6$. The modulus is $2q = 6n + 12$.

Case 1: $n = 1$ (mod 4). See Figure 44.

Step 1
Label $P_n$ starting with the first vertex with 1, 3, \ldots, $n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1, n + 1$. The remaining edge
labels are $3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n$.

**Step 2**
Label the other vertex of degree $n$ as $5n$. The corresponding edge labels are $5n + 1, 6n + 2, 5n + 3, 6n + 4, \ldots, 7n - 1, 6n$.

**Step 3**
Label the vertices of $K_4$ clockwise with $\frac{7n - 3}{2}, \frac{7n + 5}{2}, \frac{7n + 13}{2}, \frac{7n + 9}{2}$. The corresponding edge labels are $7n + 1, 7n + 9, 7n + 11, 7n + 3, 7n + 5, 7n + 7$.

![Figure 44: Properly even harmonious labeling of $DC_9 \cup K_4$ (mod 64).](image)

**Case 2: $n = 3 \pmod{4}$**. See Figure 45.

**Step 1**
Label $P_n$ starting with the first vertex with $1, 3, \ldots, n$ skipping a vertex each time. Wrap around and label the remaining vertices with $n + 2, n + 4, \ldots, 2n - 1$. Label the vertex of degree $n$ with $3n$. The edge labels corresponding to the path are $n + 3, n + 5, \ldots, 3n - 1, n + 1$. The remaining edge labels are $3n + 1, 4n + 2, 3n + 3, 4n + 4, 3n + 5, \ldots, 5n - 1, 4n$.

**Step 2**
Label the other vertex of degree $n$ as $5n$. The corresponding edge labels are $5n + 1, 6n + 2, 5n + 3, 6n + 4, \ldots, 7n - 1, 6n$.

**Step 3**
Label the vertices of $K_4$ clockwise with $\frac{7n - 1}{2}, \frac{7n + 15}{2}, \frac{7n + 3}{2}, \frac{7n + 7}{2}$. The corresponding edge labels are $7n + 7, 7n + 9, 7n + 5, 7n + 3, 7n + 1, 7n + 11$. 

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Figure 45: Properly even harmonious labeling of $DC_7 \cup K_4 \pmod{54}$. 
References


